Nonlinear System

A nonlinear system is a type of system that defies the superposition principles of homogeneity and additivity. Feedback loops between components within nonlinear systems and over time give rise to synergies or interference that make the output to the system less or more than the sum of its parts and thus nonadditive.

Nonlinear Science

Although it is often said that nonlinear systems describe the vast majority of phenomena in our world. They have unfortunately been designated as alternatives, being defined by what they are not. It might be of interest to ask why is this so? The real world we live in is inherently complex and nonlinear, but from a scientific perspective, all we have is our models to try and understand it. These models have inevitably started simple and developed to become more complex and sophisticated representations. When we say simple, in this case, we mean things that are the product of direct cause and effect interactions. With these simple interactions, we can draw a direct line between cause and effect and thus define a linear relation.

For centuries science and mathematics have been focused upon these simple linear interactions and orderly geometric forms that can be described in beautifully compact equations. Not so much because this is how the world is, but more because they are by far the easiest phenomena for us to encode in our language of mathematics and science. It is only in the past few decades that scientists have begun to approach the world of systems that are not linear. Thus, their late arrival on the scene and our lack of understanding of what they really are have lumped them with being defined by what they are not.

Additivity Principle

Linear systems are characterized by what is called the superposition principles. We can then define nonlinear systems as those the defy the superposition principles, meaning with nonlinear phenomena the principles of homogeneity and additivity break down. Additivity states that when we put two or more components together, the resulting combined system will be nothing more than a simple addition of each component’s properties in isolation.

The additivity principle, as attractively simple as it is, breaks down in nonlinear systems. Because the way we put things together and the type of things we put together affect the interactions that make the overall product of the components combination more or less than a simple additive function, and thus defies the additivity principle and we call it nonlinear. There are many examples of this such as putting two creatures together. Depending on which type of creatures we choose, we will get qualitatively different types of interaction between them. That may well make the combination non-additive. Bees and flowers create synergistic interactions or lions and deer interacting through relations of predator and prey. Both of these represent either super or sub-linear interactions.

Examples

Of course, this is all very intuitive to us. Learning about these nonlinear interactions between things is all part of growing up and learning to have a normal common sense, but the problem is actually formulating this in the language of science and mathematics. Whereas we can easily and rigorously study the properties of elements in isolation by taking them into a laboratory or some other isolated environment, it is more difficult though for us to know why, or when, or if, elements will have some special interaction. And not only this, but these interactions often create novel and surprising new phenomena through the process of emergence. Imagine you have to play the role of a matchmaker between two people you know.

You may know very well what they are like separately but it would be a lot more difficult for you to tell if they would hit it off when you introduce them to each other. To take another example, who could have imagined that when we put hydrogen and oxygen together we would get water? And not only that but because of the weak hydrogen bonds between them that this new substance would, in fact, have the property of wetness. These examples should illustrate how nonlinearity arises from the non-additive nature to the interactions between things when we combine them.

Homogeneity Principle

Next, the principle of homogeneity. That essentially states that the output to the system is always directly proportional to the input. Twice as much into the system, twice as much out, four times as much in, four times as much out and so on. The direct implication of this homogeneity principle is that things scale in a linear fashion, which clearly fails to account for the effect that the output of the previous state of the system will have on its current or future state. Put simply, our linear model does not deal with feedback loops. Inputs and outputs simply appear and disappear without any relation between them.

The homogeneity principle may often work as an approximation, but the underlying fact is that as soon as we put our system into the real world, that is to say, into an environment where it operates within both space and time, there will inevitably be feedback loops, as the actions it takes affect its environment with those effects, in turn, feeding back to affect the future state to the system. This means as soon as we start to deal with the real world, things start to get nonlinear and the more interactions we incorporate into our models – thus making them more robust and realistic – the more nonlinear things are likely to become.

Environmental Feedback

The immediate analogy that springs to mind of this is the so-called “limits to growth model.” Within the industrial age paradigm that was heavily influenced by linear systems thinking, there was or still is the belief in continuous progress without regard to the effect that the current actions of the system will have on its natural environment, and how these will feed back to affect the future input variables to the system. But as soon as we begin to conceive of this economic system within its environment, we quickly come to the conclusion that this infinite scaling is not possible because the environmental effect will inevitably feedback to constrain the future state of the system at a certain “limit to growth”. Thus, we can see how the homogeneity principle breaks down and we get nonlinear behavior. The superposition principles break down and nonlinearity arises whenever feedback interacts within a system – and feedback loops over time – are present.

Linear Systems

Linear systems are defined by their adherence to what is called the superposition principles.[1]There are just two superposition principles and they are called homogeneity and additivity. Firstly additivity, which states that we can add the effect or output of two systems together and the resulting combined system will be nothing more that the simple addition of each system’s output in isolation. So for example, if I have two horses that can each pull a hundred kilograms of weight on a cart in isolation, well if I then combine these two horses to tow a single larger cart they will be able to pull twice as much weight. Another way of stating the additivity principle is that for all linear systems, the net response caused by two or more stimuli is the sum of the response which would have been caused by each stimulus individually.[2]

The second superposition principle, homogeneity, states that the output to a linear system is always directly proportional to the input. If we put twice as much into the system we will, in turn, get out twice as much. For example, if I pay 50 dollars for a hotel room, for which I will get a certain quality of service, this principle states that if I pay twice as much I will then get an accommodation service that is twice a good. When we plot this on a graph we will see why linear systems are called linear because the result will always be a straight line.

Basic Assumptions

These principles are of course deeply intuitive to us and will appear very simple, but behind them are a basic set of assumptions about how the world works. So let’s take a closer look at these assumptions that support the theory of linear systems. Essentially what these principles are saying is that it is the properties of the system in isolation that really matter and not the way these things are put together or the nature of the relationships between them.
This is very abstract, so let’s illustrate it with some examples. Imagine you have some ailment and you have two drugs that you know are meant to cure this problem, so you take them both at the same time. The result of this, or we might say the output to this system, will depend on whether the two drugs have any effect on each other when taken in combination. If the drugs have no effect on each other then it will

be the properties of each drug in isolation that will define the overall output to the system, and because of this lack of interaction between the components our linear model will be able to fully capture and describe this phenomenon.

But if the drugs do have some effect on each other, then it will be the relationship between them that will define the system. And our linear model that does not account for this will fail as it is based upon the principle of additivity that assumes a simple additive relationship that is not the case in this situation. So we can then see the basic reasoning behind additivity, that we can simply add things without any regard to how they will interact when we put them together.

Now behind the principle of homogeneity is the assumption that scale does not matter. So say we have a business producing a million widgets a year and then scale this up to producing two million the next year. Well, maybe everything will simply scale in a linear fashion that will, of course, be captured by our linear model, but also it may not. If I can leverage economies of scale then costs will not grow in a linear fashion, and if by producing twice as much I saturate the widget market, then this will feedback to reduce my revenue resulting in a scaling that is not linear in nature.

Feedback

Linear systems models fail to capture feedback. They do not take into account the effect that an action of a system will have on its environment and how that will, in turn, feed back to affect the system again, not just in space but also in time, that is to say how past actions will feed into effect the current state of the system. The model of a linear system essentially exists in a static time vacuum.

One might wonder why we use linear systems models at all if they fail to capture so much of the real phenomena that we see in the world around us. But there are a number of good reasons why we do.

Firstly, linear models are deeply intuitive to us. The static properties of real tangible things that linear systems theory captures is much easier for us to see, touch and quantify as opposed to the intangible world of the relations between these things and over time. Secondly, as we have noted linear models do capture the behavior of some if not many systems, such as the simple interactions between particles of matter or simple dynamics of cause and effect that we might sometimes see in social and economic behavior.

Simple Systems

Lastly and probably most significantly, linear models are inherently simple. They also remove any qualitative questions surrounding the nature of the relations between elements in the system. This makes them particularly amenable to the rigorous quantitative methods of mathematics and the reductionist approach to science, where we can approach complex problems by breaking them down into their constituent parts and then tackle these simpler problems in isolation.

Synergies & Interference

A synergy is a constructive relation between two or more elements through which value is added by components performing differentiated functions, while also coordinating their activities into an integrated system. Due to the value added by the synergistic interaction, the system becomes more than the simple sum of its parts in isolation.

The term synergy comes from a Greek word that means “working together.” A synergy is a positive interaction between two elements derived from some synchronization between their states. As an example, we could cite the division of labor within many insect and human communities, such as ant colonies and market economies, where synergistic relations create a net result that is greater than the product of the individual elements actions in isolation. Very simple ants can through the division of their labor and collaboration create ant colonies that appear to far exceed the capabilities of simply summing up the capability of each ant in isolation.

Relations

A relation is a connection or interaction between two or more components. Through this connection, there is an exchange of some matter, energy, information or ideas that bind the elements into a state of interdependency, where the total gains and losses of any component are correlated with those of others in the relationship. These relationships between the system’s constituent elements can be fundamentally of two different kinds; constructive or destructive. We call constructive relations synergies and destructive relations interference.

Differentiation & Integration

From this, we should note that in order to achieve synergies, components need to be both different and synchronized. If all the ants in an ant colony or the people in a business performed exactly the same function, then the result would simply be additive. Or if they all performed different functions but did not coordinate their behavior then again we would be dealing with an additive linear system.

It is only when we get differentiation, that is when components become different not in some random fashion but in a specific way with respect to each other and they also coordinate their activities, then we get synergies and the system becomes nonlinear. To illustrate this further, say person A alone is too short to reach an apple on a tree and person B is too short as well. Once person B sits on the shoulders of person A, they are tall enough to reach the apple. The point here is that they had to both differentiate their activities with respect to each other and then coordinate them again in order to achieve this synergy. With both on the ground, they got nothing, but when one stood on the ground and the other on his shoulders then something different happened. This phenomenon of synergy is ubiquitous, being encountered throughout the natural, social and engineered world.

Interference

Whereas synergistic relations are constructive relations, we can of course also have destructive relations between components, which we might call interference. Destructive relations result in a combined system that is less than the sum of its constituent components in isolation. An example of this could be the interference between two wave functions, where they cancel each other out due to their asynchronous interaction.

Whereas synergistic relations occur through differentiation and synchronization, interference often involves a decisive lack of differentiation between components within the same environment, resulting in them trying to all access or occupy a single state with the inevitable result being destructive relations of competition and crowding out. We might think about rush hour traffic jams here, many people trying to access the same resource at the same time, with every new component added to the system resulting in an increased overall loss for everyone else due to a lack of differentiation between their activities.

Disconnection

Whereas synergies arise from feedback loops between components during their development, enabling them to adapt and synchronize their behavior with other components within the environment. Destructive relations, in contrast, involve the lack of feedback between components, meaning they cannot or do not synchronize their behavior and the overall system remains sub-linear.

For example, the human body as an entirety is a complex system that emerges out of differentiation and synchronization on both the cellular level and the level of the individual organs as they grow and are regulated through a network of feedback loops. The disease of cancer then represents a set of cells which have broken free from these feedback control mechanisms that the body exerts on all cells to regulate their proliferation. These cells grow into a malignant tumor that is then in a destructive relation with all other elements in the system.

Nonlinear Feedback Loops

A feedback loop could be defined as a channel or pathway formed by an ‘effect’ returning to its ’cause,’ and generating either more or less of the same effect. An example of this might be a dialogue between two people. What one person says now will affect what the other person will say, and that will in turn feed back as the input to what the first person will say in the future.

If we were to draw a model of a linear system, it would look something like this. There would be an input to the system, some process, and an output. As we can see, the input and the output to the system are independent of each other. The value that we input to the system now is not in any way affected by the previous output. There are of course phenomena where this holds true, such as the flipping of a coin. The value I will get from flipping a coin now will not be dependent in any way on the value I got the last time I flipped it. In mathematics, this is called the Markov property. But the fact is that many things in our world do not behave like this, meaning that current input variables to the system are dependent on previous outputs, and current outputs will affect future inputs. The state of the weather yesterday will affect the state of the weather today. The amount of money I have in my account today will through interest affect the amount I have tomorrow and so on. This phenomenon where the output of a system is “fed back” to become inputs as part of a chain of cause-and-effect that forms a circuit or loop is called a feedback loop.

Positive & Negative Feedback Loops

Feedback loops are divided into two qualitatively different types, what are called positive and negative feedback. A negative feedback loop represents a relationship of constraint and balance between two or more variables. When one variable in the system changes in a positive direction the other changes in the opposite, negative direction, thus always working to maintain the original overall combined value to the system.
An example of this might be the feedback loops that regulate the temperature of the human body. Different body organs work to maintain a constant temperature within the body by either conserving or releasing more heat. Through sweating and capillary dilation, they counter-balance the fluctuations in the external environment’s

temperature. Another example of negative feedback loops might be between the supply and demand of a product. The more demand there is for a product the more the price may go up which will in turn feedback to reduce the demand.

Nonequilibrium

We can note the direct additive relationship here. When one component goes up the other goes down in a somewhat proportional fashion, with the end result being a linear system that tends toward equilibrium. The idea of equilibrium plays a very important role in linear systems theory.

When we have these additive negative feedback loops, the net result is a zero-sum game. The total gains and losses combined are zero, and we can then define this as the system's equilibrium or "normal" state, with our models then being built on this assumption of there being an equilibrium. This concept of equilibrium holds well for isolated systems and systems in negative feedback loops, but this assumption about there being an equilibrium breaks down in nonlinear systems, and thus we describe them a being "far-from-equilibrium."

Positive Feedback

Positive feedback in contrast to negative feedback is a self-reinforcing process. The increase in the values associated with one element in the relation are correlated with an increase in the values associated with another. In other words, both elements either grow or decay together. Examples of this are numerous, such as compound interest where last year's increases result in an increase in this year's input, or chain reactions such as cattle stampedes are another example. The result is always a self-reinforcing process that leads to exponential outcomes of growth or decay. The total gains and losses are non-additive and do not sum up to zero, thus there is no equilibrium and these nonlinear systems are said to exist far-from-equilibrium. These positive feedback loops are of course unsustainable, requiring the input of energy from their environment.

The exponential growth in human industrial activity over the past few centuries could be cited here. The more developed our industrial technologies are the better we are able to process and access petroleum which again feeds back to result in more energy and more industrial development, and so on. And this is all the product of some input of energy from the system's environment that will eventually reach some limit.

Environment

As soon as we put our system into its environment, its output will in some way affect that environment. And this will, in turn, affect the future input to the system through what is called a time delay feedback loop. If the new input produces a result in the opposite direction to previous results, then it is a negative feedback and their effects will stabilize the system towards some equilibrium point. If these new inputs facilitate and accelerate the development of the system in the same direction as the preceding results, they are positive feedback resulting in nonlinear exponential growth or decay. Lastly, whereas negative feedback will lead to the system converging around some equilibrium state, positive feedback leads to divergent behavior as it rapidly moves away from an equilibrium and we describe them as being “far-from-equilibrium.”

Examples

Feedback loops are an example of the premise within complexity theory that complex phenomena can be the product of simple rules. Almost all phenomena that you would consider not normal are nonlinear. Positive feedback loops are behind a very many processes of change within complex systems. A social riot would be an example of positive feedback, when a riot begins with few people these individuals are vulnerable but with every extra person that chooses to partake in the riot it makes it more likely that it will be successful and less likely that any one individual will be reprimanded.

Thus more will beget more, as this positive feedback cascades through the individuals aligning their states. Conflict escalation can involve positive feedback, given some act of aggression an opposing agent will be threatened, becoming less tolerant and more likely to react which will in turn feedback to effect the same action on the behalf of the other. An example of this would be an arms races between two nations, where the two sides continue to try and outcompete the other leading to all losing and growing potential for conflict.

Likewise, the phenomenon of irrational exuberance is another example of positive feedback. When the value of a trader’s stock goes up this feeds back to boost the trader’s self-confidence in their decision making and encourages them to make more investments that may be even riskier. Another good example would be what is called the Matthew Effect within sociology, which describes the fact that advantage tends to beget further advantage. Thus this phenomenon is also known as the “rich get richer” as these feedback loops tend to increase initial inequalities. We might think about the fact that bank managers are more likely to lend money to people who already have lots of money.

Likewise, those who are already well connected within society will have greater potential for making more influential connections. This accumulative effect is described within network science by the concept of preferential attachment, which explains that those nodes that initially acquire more connections than others will increase their connectivity at a higher rate, and thus an initial difference in the connectivity between two nodes will increase further as the network grows.

Synchronization

Synchronization is a state of coordination between two or more elements within a system, whereby their variables are correlated in some fashion. The mathematician Steven Strogatz talks about synchronization as such. “sync is maybe one of…the most pervasive drives in all of nature. It extends from the subatomic scale to the furthest reaches of the cosmos. It’s a deep tendency towards order in nature that opposes what we’ve all been taught about entropy…(it is) the tendency towards spontaneous order. “

Any form of organization within a system is going to derive ultimately from elements assuming some coordination between their states this coordination can be understood in terms of some correlation between their state variables. There are just three different types of correlation, firstly we can have a random correlation, meaning there is no relation between the variables, they are independent, secondly we can have a positive correlation, meaning the values associated with the different elements move in the same direction, lastly we can have a negative correlation, where the two variables move in different directions.

Random Correlation

When elements or agents act completely independently, we will get a random correlation between their states, for example, me choosing to go to the swimming pool on Friday has absolutely no correlation to whether my neighbor will go shopping the following Monday, these two events are randomly correlated. Randomness is the lack of pattern or predictability in events. A random sequence of events, symbols or steps has no order and does not follow an intelligible pattern or combination. Individual random events are by definition unpredictable, but in many cases the frequency of different outcomes over a large number of events *is* predictable.

For example, if we took a large enough group of people and randomly assigned connections between them, and then went and plotted a graph of how many connections each person had, we would get a normal or Gaussian distribution, where some would have few connections, some many, but most would tend towards the average and the more connections we added the closer we could predict what this average would be, this is called the law of large number. If this is truly a random system, any individual event is perfectly random, totally unpredictable, but as we go to the limit of infinity we get an outcome that is perfectly predictable, thus the further we go towards infinity the more it will tend towards this predictable outcome.

Negative Correlations

If the values are negatively correlated we have what is called a negative causal link within system dynamics. A negative link is a relationship between two variables where they change in the opposite direction, such that as the value of one variable increases, the other decreases, and vice versa. An example of this might be the financial relationship between the owner of a business and the employees, if all other things are equal, the more the owner pays the employees the less profit for the owner, thus elements within this type of relation are going in the opposite direction creating a counterbalancing dynamic.

Positive Links

Positive causal links between values associated with different elements means they both move in the same direction. Thus if one variable increases, the other one also increases or if one decreases the other also decreases. Positive links represent relations of deep interdependency, everyone wins or losses together. We get this positive link by two things interacting in either a constructive fashion meaning they both increase or a destructive fashion meaning they both decrease.

By constructive we mean the two variables are moving in the positive direction, by two countries signing a trade agreement their two economies may grow and this is essentially a synergistic interaction, by both increasing together they are adding value to the overall system. Inversely with a destructive interaction the two variables move in the downward direction together an arms race between to nations might be an example of this, through this interaction the value of the whole system is decreasing, this may be called a negative synergy or interference in the way that two sound waves can cancel each other out making the combined output to the system less than a simple summation of its inputs.

Positive links are nonlinear, when we add and subtract the gains and losses to all agents in the interaction they do not sum to zero thus defying the additivity principle. This nonlinear nature to positive links makes these phenomena less well studied and understood. Wherever we have synergies or interference within a system we will get nonlinearity, this might be the synergy between two partners in a relationship, between to businesses engaged in a merger, different drugs taken at the same time or between to creatures within an ecosystem, these nonlinear phenomena are ubiquitous in our world and form the foundation for our understanding of synchronizations.

Self-Organizing Criticality

Self-organized criticality (SOC) is a property of nonlinear dynamical systems that have a critical point as an attractor, meaning the system endogenously organizes itself into a critical state. Phenomena that are a product of SOC exhibit a power law distribution in the frequency of their occurrence. This phenomenon of self-organized criticality has been identified in many different systems from earthquakes to fluctuations within financial markets, to ecological evolution to outbreaks of epidemics and the occurrence of solar flares. Self-organized criticality is typically illustrated with reference to what is called the sandpile model, developed by researcher Pear Bac. The sandpile model was the first model to exhibit self-organized critical behavior, where the system endogenously moves towards its critical (phase transition) point.

Sandpile Model

The sandpile model is taken from the empirical observation that when we drop small grains (of something like sand) on top of each other they will build up into a pile with occasional grains running off, one or two at a time, in proportion to the rate at which we are dropping them, this is the linear equilibrium state to the system's development, grains of sand are held on the pile by its low incline and the friction from other grains that have already built up (this is the attractor). But at some critical point the pile of sand has built up to such an extent that the incline on the side has reached a critical level, by dropping just one additional grain of sand we can cause a cascading avalanche, a positive feedback as each new grain of sand that cascades down destabilizes the system more which will feedback to effect more grains to slide off.

The sandpile phenomenon is a classical example of nonlinear change, here we can note the prolonged period for which the system was held in a stable equilibrium and the very short period of rapid nonlinear change, this is also an illustration of the idea of punctuated equilibrium; prolonged periods of stability and then rapid phase transitions characteristic of nonlinear change.

The avalanche is a product of positive feedback the more grains that are falling the more likely they will display additional grains that will then augment the size of the pile and so on a positive feedback loop leading to an avalanche. Thus these self-organized systems like the sand pile are nonlinear in that a small perturbation to the system can have a very negligible effect or it can have a radically disproportionate one where a single grain inputted can cause an entire avalanche, we do not know when it will either.

Power Laws

The size of the avalanche and the number of times it occurs represents a power law distribution, meaning there will very many, very small slides and very flew very large slides, this power law distribution that is a common feature within nonlinear systems allows for events that are statistically virtually impossible within linear system, these very large events are called “black swan” events. Many phenomena exhibit this power law distribution including stock market crashes. Some stock market crashes are so large that they would be virtually impossible given a normal distribution, and this is coming from the positive feedback, of herd behavior in this case.

Self-Organization

To illustrate the self-organizing dimension to SOC we could think about the tragedy of the commons as an example of SOC within social systems. Where out of local events – that is to say by everyone acting in their own rational self-interest, using the commons as much as possible – this will drive the whole system to a critical state where we get the overuse of the commons and global collapse. The point being that this collapse was the attractor to that system. Thus we can say that it is the way that the local rules are setup that creates the destructive global outcome.

Exponents & Power Laws

The term power law describes a functional relationship between two quantities, where one quantity varies as a power – or exponent – of another. Part of the definition to linear systems is that the relationship between input and output is related in a linear fashion. The ratio of input to output might be 2 times as much, 10 times as much or even a thousand times. It is not important, as this is merely a quantitative difference. What is important is that this ratio between input and output is itself not increasing or decreasing. But with feedback loops the previous state to the system feeds back to affect the current state, thus enabling the current ratio between input and output to be greater or less than its ratio previously, and this is qualitatively different.

Exponents

This phenomenon is captured within mathematical notation as an exponential. The exponential symbol describes how we take a variable and we multiply it by another,

not just once but we in fact iterate on this process, meaning we take that output and feed it back in to compute the next output. Thus, the amount we are increasing by each time itself increases.
An example of this might be the growth of bacteria when given the right conditions. If I wish to create a test tube of bacteria, knowing that the bacteria will double every second, I start out in the morning with only two bacteria hoping to have my tube full by noon. As we know the bacteria will grow exponentially as the population at any time will feed into effect the population at the next second, like a snowball rolling down a hill. It will take a number of hours before our tube is just 3% full but within the next five seconds as it approaches noon it will increase to 100% percent of the tube. This type of disproportional change with respect to time is very much counter to our intuition where we often create a vision of the future as a linear progression of the present and past. The important thing to note here is that in exponential growth the rate of growth itself is growing, and this only happens in nonlinear systems where they can both grow and decay at an exponential rate.

Power Laws

Exponentials are also called powers and the term power law describes a functional relationship between two quantities, where one quantity varies as a power of another. There are lots of examples of the power law in action but maybe the simplest is the scaling relationship of an object like a cube. A cube with sides of length A will have a volume of A3, and thus the actual number that we are multiplying the volume by grows each time. This would not be the case if there was a simple linear scaling such as the volume being 2 times the side length.
Another example from biology is the nonlinear scaling within the metabolic rate vs. size of mammals. The metabolic rate is basically how much energy one needs per day to stay alive, and it scales relative to the animal’s mass in a sub-linear fashion. If you double the size of the organism then you actually only need 75 % more energy. One last example of the power law will help to illustrate how it is the relationships between components within a system that is a key source of this nonlinearity.

Network Effect

The so-called Metcalfe’s Law comes from the world of I.T. and it derives from the simple observation that the number of possible cross connection in a network grows as a square of the number of computers in the network. Every time we add a new computer to the network we have the possibility of adding as many more links as there are computers in the network. So whereas the number of computers grows in a linear fashion the number of links can grow in a super linear or exponential fashion. As each person who joins the network makes it more valuable, Metcalfe’s law leads to the value or power of a network increasing in proportion to the square of the number of nodes on the network. This is of course not restricted to just computer networks, but is a feature of all networks and thus is given the more general name the network effect. The network effect is a key driver of positive feedback as every time someone links to a particular node on a network it makes it that bit more likely someone else will also. This example helps to illustrate the dynamics behind positive feedback and how through these positive feedback loops the system can move or develop in a particular direction very rapidly.

Many real-world networks such as the World Wide Web have proven to have this power law relationship between the size and quantity, where there are just a very few sites with a very large size and very many of a very small size. We should note here again that with the network effect as with all nonlinear systems, things can go both ways. It may have helped to grow the internet to its vast size in a very short period of time which we might cite as a positive thing, but also the network effect is in operation when some negative news about your company goes viral and behind the creation of herd mentalities. Exponentials are such a powerful force for change because the system is not just growing or decreasing, but that due to the feedback loops and synergies within the system and over time there is also another meta-level to the system's development that is itself increasing this rate of growth or decay to enable very rapid change.

Long Tail Distributions

A long tail distribution is one that has very few occurrences of very large events, and very many occurrences of very small events, which gives the graph a "long tail." This "long tail" is generated by a power law relationship, meaning there is a power relationship between the size of an event and the likelihood of its occurrence.[1] In statistics, a long tail of some distribution of numbers is the portion of the distribution having a large number of occurrences far from the "head" or central part of the distribution. They are typical of nonlinear systems, as nonlinear systems are "non-normal", meaning there is statistically no normal average or typical state within the system.

With linear systems, there is some kind of mean state to the system. Given enough samples of the different states within the system, we will be able to compute some average that we can use as a representative of the whole system. For example, the distribution of people's height, if we were to plot them, would follow what is called a normal distribution, meaning there will be very many people around the average of say 5 to 6 feet, some a bit larger and some a bit smaller than this, say between 3 and 8 feet, but virtually no one outside of this range of states. These normal distributions then have a well-defined center and then drop off exponentially fast, meaning there is an extraordinarily low probability of getting extreme states.

Normal Distributions

A feature to normal distributions is the so-called law of large numbers, which means that the average of the results obtained from a large number of trials should be close to the expected value, and will tend to become closer as more trials are performed. This so-called law is important because it “guarantees” stable long-term results for the averages of some random events. For example, while a casino may lose money in a single spin of the roulette wheel, its earnings will tend towards a predictable percentage over a large number of spins. Any winning streak by a player will eventually be overcome by the law of large numbers that will bring the net gains and losses back to an average.

Non-Normal

The normal distribution holds for many linear systems, physical and chemical, but in the world of non-linearity, the idea that there is such a thing as normal and we should expect this normal is no longer applicable. The power law nature of nonlinear systems creates what is called a long tail distribution, meaning extraordinary events that are virtually impossible within normal distributions are possible within nonlinear systems. These extraordinary events are referred to as “black swans”.

For example on October 19th, 1987 on Black Monday, the Dow Jones stock market index dropped by 22% in a single day. Compared to the typical fluctuation of less than 1% this was a shift in more than 20 standard deviations from the norm. Within a normal distribution, this would be virtually impossible, with something like a 10 to the power of 50 chance of it happening.

These extraordinary events can then have a dramatic effect on the system’s average behavior because if we take a random sample we will get a certain average but then if we add just one more node to this it might be a black swan that will radically alter the average again. In the example of measuring people’s heights, if we have a room of people and then the tallest person in the world walks in, the average height will only change by a few feet. But say we are measuring people’s income and now the richest person in the world walked in with an income of many millions, this would so radically alter the average income for it to become nonsensical.

In a power law distribution as we increase the number of samples we take, values will not converge to an average. They will, in fact, diverge, with some exceptions. Asking for an average is like asking how big is a stone or how long is an average piece of string?

Interconnections

The normal distribution is largely derived from the fact that we are taking random samples from components that have no correlation between them. If I flip a coin now, it will not affect what I get on the next flip and will thus follow a normal distribution. If one person wins on one casino table, it will not affect whether someone else will win on another and so on. But in nonlinear systems, things are arranged in a particular way. Large websites are large because of the network effect and because people have specifically chosen to connect to them. There is nothing random about this. Financial crashes are also similar in nature.

Bifurcations & Attractors

An attractor is a set of states towards which a system will naturally gravitate and remain cycling through unless perturbed. For any given system we can create a state space representing all the possible states that the system might take, the attractor is then a subset of those states that correspond to the system's typical behavior. A bifurcation is a qualitative topological transformation in this state space resulting in a spitting of this attractor into two distinct stable attractors.[2]

State Space

A state space – also called a phase space – is a mathematical model in which all possible states of a system are represented, with each possible state of the system corresponding to one unique point in the phase space. In order to build this state space model, we have to define one or more parameters to the system that we are interested in, where a parameter is simply a measurement of something about the system.

If we were interested in a sales person's finances we could define a parameter, to measure their income, but this would not be very interesting it would simply go up and down depending on their sales, so what we are typically interested in then is the relationship between two or more different parameters. We might define another parameter to their overall savings or wealth. Now at each day we will take a sample of both of these parameters, putting a dot at the corresponding value and stay doing this over a period of time. What we will see after doing this for a few weeks is some kind of typical behavior, during the week they are earning some amount, then it goes up on Saturday with lots of sales but then down on Sunday when they are not working and then starts again the next week. What we will typically see is that these different states do not go around every single state in the whole space but are confined to a limited subset of all the possible states. This subset of the phase space of the dynamical system corresponding to its typical behavior is the attractor.

Basin of Attraction

A bowl containing a ball may be used to illustrate the concept of a basin of attraction. The ball will move around the bowl until eventually, it comes to rest at the lowest point. We can say that it is 'attracted' to that point, i.e each part of the bowl can be regarded as leading to that stationary point, and the whole bowl is what we call the system's basin of attraction.

Systems, like this ball, are typically held within their attractor because of the different forces placed upon them by their environment. An animal stays on a particular patch of fertile land and does not stray too far from it because it needs to eat, a person gets up and goes to work every day because they need the money to support themselves. What is happening is that these dynamical systems are dissipative, meaning they need some source of energy to maintain their dynamic state, they are continuously inputting new energy and then dissipating it, and they cycle through this process always having to come back to the source of energy that is maintaining their dynamic state, and it is in that cycling that we get all the different states within the attractor.

Example

For example, an attractor may represent a social institution of some kind, social institutions serve some function for individuals and society, they are essentially patterns of behaviour or belief that exist within a given society in order to serve basic human functions. Institutions represent pre-existing solutions to given social challenges both personal and social, as such they are the course of least resistance for individuals within that society, working for an existing company is typically easier than creating one's own, adopting the values of one's society is typically much easier than reading a big pile of philosophy books to figure out one's own beliefs and values. These attractors then keep social actors within a well-defined set of behaviours and some equilibrium state.

Bifurcation

The word "bifurcation" means splitting or cutting in two. If a river divides into two smaller streams, that is a bifurcation. If you split a company into two divisions, that is a bifurcation too. Mathematicians have borrowed the term bifurcation to describe

how a system branches off into a new qualitatively different long-term state of behaviour. What we are interested in complex systems is primarily a bifurcation within an attractor, meaning instead of having just one attractor in the state space, a bifurcation will now give us two attractors and that means two stable sets of states that the system may cycle through.
We could cite the French Revolution as an example, in particular what is called the tennis court oath which was a pivotal event during the first days of the French Revolution, when the Third Estate, after being locked out from the government, made a makeshift conference room inside a nearby tennis court, calling themselves the National Assembly they went on to form the new political republic of France. Prior to this event, we had a single attractor within the political state space to the nation, it was an absolute monarchy all political activity was beneath and in relation to the monarch, this tennis court oath was then a bifurcation in the topology as a new attractor formed.

Punctuated Equilibrium

Punctuated equilibrium is a model first derived from evolutionary biology. This model deals with the dynamics of a complex system, suggesting that most nonlinear systems exist in an extended period of stasis, which is later punctuated by sudden shifts of radical change. Complex systems are characterized by long periods of stability where negative feedback loops work to maintain an equilibrium holding them within a well-structured attractor state, this is then punctuated by large—though less frequent—shifts, driven by a positive feedback loop that drives the system far-from-equilibrium and out of its current attractor into a new one, during a phase transition that represents a new regime and new equilibrium, under a new set of negative feedback loops.
Whereas negative feedback loops lead to an equilibrium state and a stable linear state of development, where the input and output ratio to the system stays constant leading to an incremental linear progression, real world complex systems like ecosystems and economies are a whole network of both positive and negative feedback loops operating on many different levels from the micro to the macro. The overall state that the system exhibits is a product of the balance between these two. Negative feedback is holding it in its current configuration. Positive feedback is always trying to drive it out of this equilibrium.

For example, as long as one stays getting up every day and going to work, one will be able to remain in one’s current financial state of stable development. But when you do not go to work and stay at home playing computer games all day, this

negative feedback loop will become broken. There is now a broken symmetry between what is being earnt and the expenses, and the longer ones stays out of work the further away one is moving from this equilibrium, possibly resulting in a regime shift as the person becomes permanently unemployed.
This broken symmetry and runaway positive feedback lead to punctuated equilibrium. We get a symmetry breaking and the system moves far from its equilibrium into a phase transition as it moves into a new regime, with a new attractor and new equilibrium, and once again a new set of negative feedback loops taking over. The result is this punctuated equilibrium that is a product of an interplay between negative and positive feedback development.

Path Dependence

This dynamic to nonlinear systems creates path dependency which explains how the set of states to a system now are limited and defined by the historical trajectory that led to this point in time. That is to say, complex systems bare their history on their shoulders. Time reversibility only holds for some linear systems, but nonlinear systems are non-time reversible, the development of the system goes in one direction with respect to time, because of feedback loops, the system is within a particular attractor due to the choices made in the past.
An example of this we could cite might be the clustering of businesses. Similar businesses tend to congregate together geographically; opening nearby similar companies that attract workers with expertise in that business domain, this then draws in more businesses in search of experienced workers. This network effect is driven by positive feedback loops and negative externalities that have taken the system down a particular pathway into a particular basin of attraction from which it would be difficult to exit or alter.

Phase Transitions

Phase transitions and bifurcations are periods of qualitative and often rapid change in the dynamics of a nonlinear system's state. They involve positive feedback loops that drive the system far-from-equilibrium and result in exponential change and a pattern of development called punctuated equilibrium where periods of stability are punctuated by phases of rapid change called the phase transition period.

Feedback Loops

The qualitative dynamic behavior of nonlinear systems is largely defined by the positive and negative feedback loops that regulate their development, with negative feedback working to dampen down or constrain change to a linear progression, while positive feedback works to amplify change typically in a super-linear fashion. As opposed to negative feedback where we get a gradual and often stable development over a prolonged period of time – what we might call a normal or equilibrium state of development – positive feedback is characteristic of a system in a state of nonequilibrium.

Positive feedback development is fundamentally unsustainable because all systems, in reality, exist in an environment that will ultimately place a limit on this grown. From this we can see how the exponential grow enabled by positive feedback loops is what we might say special. It can only exist for a relatively brief period of time. When we look around us, we see the vast majority of things are in a stable configuration constrained by some negative feedback loop, whether this is the law of gravity, predator-prey dynamics or the economic laws of having to get out of bed and go to work every day. These special periods of positive feedback development are characteristic and a key driver of what we call phase transitions.

Transition

A phase transition may be defined as some smooth, small change in a quantitative input variable that results in a qualitative change in the system's state. The transition of ice to steam is one example of a phase transition. At some critical temperature, a small change in the system's input temperature value results in a systemic change in the substance after which it is governed by a new set of parameters and properties. For example, we can talk about cracking ice but not water, or we can talk about the viscosity of a liquid but not a gas, as these are in different phases under different physical regimes. And thus, we describe them with respect to different parameters.

Another example of a phase transition may be the changes within a colony of bacteria, that when we change the heat and nutrient input to the system we change the local interactions between the bacteria and get a new emergent structure to the colony. Although this change in input value may only be a linear progression, it resulted in a qualitatively different pattern emerging on the macro-level of the colony. It is not simply that a new order or structure has emerged but the actual rules that govern the system change. And thus, we use the word regime and talk about it as a regime shift, as some small change in a parameter that affected the system on the local level leads to different emergent structures that then feedback to define a different regime that the elements now have to operate under.

Examples

Examples of phase transitions might include the fall of the Berlin Wall, before this rapid critical phase transition the global political environment was largely defined by a bipolar regime. Before the fall this bipolar model was the parameter we used to define the system, after the event, the political environment was described with reference to a new set of parameters relating to globalization.
The Arab Spring might be another example. The Arab Spring is widely believed to have been instigated by dissatisfaction with the rule of local governments. After many decades of the Middle East being held within a particular configuration or political regime, the Arab Spring was a punctuation of this equilibrium. The previous regime was a set of negative feedback loops that balanced the system into some equilibrium – we might say there was some balance of power – but this balance got broken through some small fluctuation, the self-sacrifice of a street vendor in Tunisia, this small event then got amplified by positive feedback into a large systemic transformation. Through this positive feedback, the balance of power was broken temporarily and the political system across the Middle East moved into a phase transition.

Bifurcation Theory

Another way of talking about this is in the language of bifurcation theory. Whereas with phase transitions we are talking about qualitative changes in the properties of the system, bifurcation theory really talks about how a small change in a parameter can cause a topological change in a system's environment, resulting in new attractor states emerging. A bifurcation means a branching. In this case, we are talking about a point where the future trajectory of an element in the system divides or branches out as new attractor states emerge. From this critical point, it can go in two different trajectories which are the product of these attractors. Each branch represents a trajectory into a new basin of attraction with a new regime and equilibrium.

To take a real world example of a bifurcation, say you have been studying Fine Art as an undergraduate. This subject has for the past few years represented your basin of attraction, that is to say, your studies have cycled through its many different domains but never moved off into another totally different subject. But now that you have graduated, you have the option to continue your studies in either sculpture or painting. You have now reached a bifurcation point as two new attractors have opened up in front of you. Some small event at this point could define your long-term trajectory into one of these two different basins of attraction.

Punctuated Equilibrium

As opposed to linear systems that may develop in an overall incremental fashion, the exponential growth that nonlinear systems are capable of -through feedback loops and phase transitions – leads to a different overall pattern to their development, what we might call punctuated equilibrium. Within this model of punctuated equilibrium, the development of a nonlinear system is marked by a dynamic between positive and negative feedback, with negative feedback holding the system within a basin of attraction that represents periods of stable development. These stable periods are then punctuated by periods of positive feedback which takes the system far from its equilibrium and into a phase transition as the fundamental topology of its attractor states change and bifurcate.

Examples of this punctuated equilibrium might be the development of economies that go through periods of stable growth then rapid change through an economic crisis and recovery, or ecosystems as they collapse due to some environmental change and then an ensuing period of rapid re-growth towards a new equilibrium of stable development again. The same punctuated development may be seen within the development of a human being as they go from childhood to adulthood to old age, each period representing a stable basin of attraction with changes between each being marked by periods of rapid and defining change.

Butterfly Effect

The butterfly effect is a popular term for what is called sensitive dependence on initial conditions in which a small change in some input state to a nonlinear system can result in a vastly disproportionate output at a later stage. The term is thought to derive from the title of a talk given by Edward Lorenz in 1972 called “Does the flap of a butterfly’s wings in Brazil set off a tornado in Texas?” The butterfly effect is part of the broader area of chaos theory that studies the dynamics of nonlinear systems (in particular those that are sensitive to initial conditions). After having gained acceptance within mainstream mathematics and science during the seventies and eighties, the butterfly effect has been since identified in many different areas from weather patterns, to supply chains networks, to electrical power grids and international politics.

Cause & Effect

Everything has an effect on everything else in our universe. Every single particle of matter has some gravitational effect on every other particle of matter. Irrespective of how much we try to isolate something, the concept of an isolated system that has zero interaction with its environment is really just a theoretical one that does not exist in reality.

This simple insight literally destroys the entire scientific enterprise. Our traditional conception of science is dependent upon this capacity to isolate systems, because if we want to say A causes B then we need to be able to control all other variables, that is remove them from the equation. As we have stated this is not possible. So how do we get around this stumbling block, as it appears the scientific endeavor goes on without this causing too much of a problem? What we do, because we can not fully isolate any system, is essentially define what are significant effects and what are negligible effects and simply forget about the negligible effects. For example, if I am doing some research in my lab on the interaction of two particles of matter, under this premise I do not need to take into account the gravitational effect that some planet on the other side of the universe is having on these particles because it is deemed negligible.

This basic premise that small causes can only cause relatively small effects is one of the basic assumptions and principles that gives our world some order. We depend on it almost all day, every day. I feel confident in the fact that if I forget to pay my bank overdraft this week, it is not going to bring the whole global economy into meltdown, or that a teeny little pin prick is not going to kill me. We find order in the world in the fact that the chances of these phenomena happening are so small, that they are negligible and we can thus forget about them. Without this being the case within linear systems our world would be extraordinarily random and chaotic.

Feedback Loops

In nonlinear systems, though, feedback loops can grow exponentially. This means that negligible effects or differences within nonlinear systems can themselves grow in an exponential fashion where small effects and errors are fed back into the system at each stage of its development to compound the size of the errors as it grows in an exponential fashion.

As was the case in the famous Edward Lorenz computer experiment, where when he fed values into the computer that he thought were exactly the same, the output results the computer gave him were widely divergent. He eventually traced this back to small differences in rounding errors that made the values only very slightly different. But through iteration, these very small errors would grow not in a linear fashion but exponentially, making the resulting output widely divergent within a relatively short period of time and thus giving us the phenomenon that is called sensitivity to initial conditions.

Sensitivity to initial conditions is popularly known as the “butterfly effect,” thought to be so called because of the title of a talk given by Edward Lorenz in 1972 called “Does the Flap of a Butterfly’s Wings in Brazil set off a Tornado in Texas?” The flapping wing represents a small change in the initial condition of the system, which causes a chain of events leading to some large-scale phenomena. Had the butterfly not flapped its wings, the future trajectory of the system might have been vastly different. Something to note is that the butterfly does not directly cause the tornado. This is, of course, impossible. The flap of the butterfly’s wings simply defines some initial condition. It is then the set of chain reactions through feedback loops that enable a small change in the initial conditions of the system to have a significant effect on its output, rendering long-term predictions virtually impossible.

Nonlinearity

With respect to the unpredictable nature of the butterfly effect, one might say, well if our initial measurement is wrong then obviously our prediction of its future state is also going to be wrong! But this is missing the point, which is firstly that this inaccuracy is growing exponentially as the system develops – it is not just staying the same, and secondly that in these nonlinear systems we can never know exactly the starting condition as our accuracy of measurement must grow in an exponential fashion in order to achieve just a linear growth in our horizon of predictability. Chaos and the butterfly effect after being shunned by the scientific community for many decades are today accepted as scientific facts, a fundamental and inescapable part of the dynamics of nonlinear systems. They show again how when things can grow exponentially we can get extraordinary and counter-intuitive results.

Dynamical Systems

Within science and mathematics, dynamics is the study of how things change with respect to time. As opposed to describing things simply in terms of their static properties, the patterns we observe all around us in how the state of things changes over time is an alternative way through which we can describe the phenomena we see in our world.

A state space – also called a phase space – is a model used within dynamical systems to capture this change in a system's state over time. A state space of a dynamical system is a two or possibly three-dimensional graph in which all possible states of a system are represented, with each possible state of the system corresponding to one unique point in the state space.

Continuous & Discrete Models

It is possible to model the change in a system's state in two ways, as continuous or discrete. Firstly, with continuous models, the time interval between our measurements is negligibly small making it appear as one long continuum, and this is done through the language of calculus. Calculus and differential equations have formed a key part of the language of modern science since the days of Newton and Leibniz. Differential equations are great for few elements. They give us lots of information, but they also become very complicated very quickly.

On the other hand, we can measure time as discrete meaning there is a discernible time interval between each measurement, and we use what are called iterative maps to do this. Iterative maps give us less information but are much simpler and better suited to dealing with very many entities where feedback is important. Whereas differential equations are central to modern science, iterative maps are central to the study of nonlinear systems and their dynamics as they allow us to take the output of the previous state of the system and feed it back into the next iteration, thus making them well designed to capture the feedback characteristic of nonlinear systems.

Transience Motion

The first type of motion we might encounter is simple transient motion, that is to say, some system that gravitates towards a stable equilibrium and then stays there. Such as putting a ball in a bowl, it will roll around for a short period before it settles at the point of least potential gravity, its so-called equilibrium, and then will just stay there until perturbed by some external force.

Periodic Motion

Another common type of motion we encounter is periodic motion. For example, the motion of the planets around the Sun is periodic. This type of periodic motion is of course very predictable. We can predict far out into the future and way back into the past when eclipses happen. In these systems, small disturbances are often rectified and do not increase to alter the system’s trajectory very much in the long run. The rising and receding motion of the tides or the change in traffic lights are also examples of periodic motion. Whereas in our first type of motion the system simply moves towards its equilibrium point, in this second periodic motion it is more like it is cycling around some equilibrium.

Dissipation

All dynamic systems require some input of energy to drive them. In physics, they are referred to as dissipative systems as they are constantly dissipating the energy being inputted to the system in the form of motion or change. A system in this periodic motion is bound to its source of energy, and its trajectory follows some periodic motion around it, or towards and away from it.

For example, the human body requires the input of food on a periodic basis, we consume food then dissipate it through some activity and then consume more and dissipate it again in a somewhat periodic fashion. Like other biological systems, we are bound to cycle through this set of states. The same is true for our car or a business that are constrained by the inputs of fuel or finance. The dissipation and the driving force tend to balance, setting the system into its typical behavior. This typical set of states the system follows around its point of equilibrium is called an attractor.

Attractors

In the field of dynamical systems, an attractor is a set of values or states toward which a system tends to evolve for a wide variety of starting conditions to the system. System values that get close enough to the attractor remain close even if slightly disturbed. There are many examples of attractors such as the use of addictive substances. While being subject to the addiction our body cycles in and out of its physiological influence but continuously comes back to it in a somewhat periodic and predictable fashion, that is, until it is able to break free from it.

A so-called basin of attraction then describes all the points within a state space that will move the system towards a particular attractor. For example, a planet’s gravitational field is a basin of attraction. If we place some matter that is large enough into the gravitational field, it will be drawn into its orbit irrespective of its starting condition. An attractor can be thought of as a subspace of a state space that closes in on itself.

Nonlinear Dynamics

Transience and periodic motion are characteristic of linear systems, relatively simple systems with a single point equilibrium. But the dynamics of nonlinear systems involve the interaction of more than just one attractor, and they may have multiple equilibria, with their long-term trajectory being sensitive to initial conditions making it virtually impossible to predict with any accuracy. This is what is called chaos; sensitive dependence upon initial conditions that allows the behavior of a deterministic nonlinear system to be non-predictable.

Nonlinear Dynamics & Chaos

Chaos theory is the study of nonlinear systems dynamics whose development is marked by iteration and feedback loops, making them sensitive to initial conditions (what is called the butterfly effect). Even though these nonlinear systems may be deterministic, such as a double pendulum, their long-term trajectory is typically not possible to predict without running the system.

Linear & Nonlinear

Isolated systems tend to evolve towards a single equilibrium, a special state that has been the focus of many body research for centuries. But when we look around us, we do not see simple periodic patterns everywhere. The world is a bit more complex than this and behind this complexity is the fact that the dynamics of a system may be the product of multiple different interacting forces, have multiple attractor states and be able to change between different attractors over time.

A classical example given of this is a double pendulum. A simple pendulum without a joint will follow the periodic and deterministic motion characteristic of linear systems with a single equilibrium. Now if we take this pendulum and put a joint in the middle of its arm so that it has two limbs instead of one, now the dynamical state of the system will be a product of these two parts’ interaction over time and we will get a nonlinear dynamic system. To take a second example, the dynamics of a planet orbiting another is an example of a linear system with a single equilibrium and attractor, but when we add another planet into this equation, we now have two equilibrium points creating a nonlinear dynamic system as our planet would be under the influence of two different gravitational fields of attraction.

Sensitivity to Initial Conditions

Whereas with simple periodic motion it was not important where the system started out, there was only one basin of attraction and it would simply gravitate towards this equilibrium point and then continue in a periodic fashion. But when we have multiple interacting parts and basins of attraction, small changes in the initial state to the system can lead to very different long-term trajectories and this is what is called chaos. Wikipedia has a good definition for chaos theory: "Chaos theory studies the behavior of dynamical systems that are highly sensitive to initial conditions—a response popularly referred to as the butterfly effect. Small differences in initial conditions yield widely diverging outcomes for such dynamical systems, rendering long-term prediction impossible in general."

Unpredictable Nature

We should note that chaos theory really deals with deterministic systems, and moreover it is primarily focused on simple systems in that it often deals with systems that have only a very few elements, as opposed to complex systems where we have very many components that are nondeterministic. In these complex systems, we would, of course, expect all sorts of random, complex and chaotic behavior, but it is not something we would expect in simple deterministic systems.

This chaotic and unpredictable behavior happens even though these systems are deterministic, meaning that their future behavior is fully determined by their initial conditions, with no random elements involved. In other words, the deterministic nature of these systems does not make them predictable, which is deeply counter-intuitive to us. A double pendulum essentially consists of only two interacting components – that is each limb – and these limbs are both strictly deterministic when taken in isolation. But when we join them, this very simple system can and does exhibit nonlinear and chaotic behavior. In these chaotic systems, their unpredictable nature emerges out of the interaction between their components.

Complex Chaotic Systems

Although chaos theory deals with simple nonlinear systems, the phenomenon of sensitivity to initial conditions is also a part of complex systems as we might expect. For example, say I am walking to the subway station on my way to work but as I pass the bus stop I notice a bus just pulling in that I recognize as one that will take me near to where I want to go. So I jump on the bus and it takes me off into a different basin of attraction than if I had arrived just 30 seconds later. In complex systems this sensitivity to initial conditions can be very acute during particular stages in their development, what are called phase transitions. When they are far-from-equilibrium, small fluctuations can push them into new basins of attraction.

Fractals

Fractals are both physical phenomenon and mathematical objects, exhibiting a repeating pattern that is displayed at various levels of magnitude, what is called scale invariance, creating self-similar patterns under magnification. Fractals are sometimes called the pictures of nature, and if chaos theory, as the name implies, is the chaotic and unpredictable dimension to nonlinear systems, then we might say fractals represent their orderly side.

We see lots of order in our world. The Earth goes around the Sun today and it will do the same tomorrow and the next day for millions of years. If we take a butterfly, we see that one side of it is almost exactly mirrored in the other. We see the same in the regular geometric forms of snowflakes. One way of understanding this order is through the concept of symmetry. In mathematics, symmetry can be defined as an object or process that is invariant to a transformation. In more familiar terms, this means something that stays the same despite a change. With the example of the butterfly, its external physiology had a reflection symmetry, meaning we can simply flip one side over and we will get the other. The same is true for the snowflake, and we could imagine some symmetry in the pattern of earth's orbit.

Symmetry & Order

Without these symmetries to the world, the scientific endeavor would be very difficult because science is about the creation of compact representations of the world. It is like we are creating maps of the world and it is only through finding these symmetries and encoding them in models that we can describe a wide variety of phenomena with simple equations. Without these symmetries, our scientific map of the world would have to be the same size as the world itself. Symmetry and asymmetry are then two of the most powerful concepts in mathematics and science for talking about order and chaos.

They help us to understand these abstract concepts in a distinctly geometric and visual form. With the phenomena of chaos, we see a breaking of symmetry. Two things that started out similar became increasingly dissimilar. The symmetry between them became broken, and the result was after a short period of time complete asymmetry. Fractals have what is called scale invariance, that is, they have a symmetry with respect to scale, meaning the scale can change but the structure will repeat itself over various levels of magnitude. This scale invariance is also called self-similarity, and it is this type of symmetry under magnification that gives fractals an amazing type of structure and order. Fractals are both mathematical constructs that derive out of iterative functions and real world phenomena.

Iteration

As with most of nonlinear systems, the core ideas behind fractals is of feedback and iteration. The creation of most fractals involves applying some simple rule to a set of geometric shapes or numbers and then repeating the process on the result. For example, the most famous fractal called the Mandelbrot set, so named after the discoverer of the concept of fractals, is a product of a simple iterative map on complex numbers. We will not go into the details of complex numbers, but the iterative map itself is quite simple. There are many extraordinary things about fractals but the first thing we will note is the infinite variety that these simple iterative functions can produce.

Infinite Detail

Whereas most Euclidian shapes, what we might call normal shapes, tend towards a bland featurelessness as we scale them up, if we take something like a circle the more we zoom in on it the more it will start to appear like a bland straight line and this would also be the case for other regular shapes such as squares, triangles and so on. The iterative functions behind fractals, in contrast, give us an infinite amount of structure and detail within their form. Irrespective of if we divide the shape into two or divide it into a million, each of these million little parts will itself have an infinite amount of detail and a form that resembles the whole.

Infinite Length

Just as there can exist an infinite amount of form in a finite object there can with fractals also exist an infinite length within any finite length. This is best demonstrated by a fractal called the Koch curve that can be obtained by iterating a simple process of dividing a line and placing a triangle in its center, and then iterating on this to divide the lines on the triangle in a similar fashion. What we get when we do this is a path that will, in fact, have an infinite length to it. If we were to try and trace a path along this line, its infinite detail would prevent us from ever reaching the end. Thus, this finite length contains an infinite length within it.

Power Laws

The last thing we will note about fractals is that their scale-free property means that no scale is the "proper" frame of reference. Most linear systems that represent regular forms have features and structures within a limited range of scales. Thus, if we plot them we get a normal distribution, with the majority of features tending towards a mean, giving it some kind of "normal" frame of reference with respect to scale.

These nonlinear fractals as we have noted, have scale invariance, meaning we will find features on all levels, resulting in there being no normal distribution to their occurrence and thus no normal frame of reference. The occurrence of features is instead distributed out as a power law. Big features exist but are very rare, whilst many small features also exist and this long tail does not drop off – because it is a fractal – it will go on infinitely. Thus, we can say that power laws are the algebraic expression of the scale-free structure to fractals. And just as there is no "normal" scale in fractals, there is no normal average to power law distributions.