# Network Theory

## Network Paradigm

All models and theories are like windows onto the world. None of them are perfect. They all enable us to see some things but also inhibit us from seeing others, and more importantly they all rest upon some set of assumptions. This set of methods and assumptions that support a particular scientific domain is called a paradigm, and network theory is based upon a paradigm that has a number of features to it that we can identify.

## Connectivity

Firstly, networks are all about connectivity. Within systems whose components are relatively isolated, we can focus our interest on the individual components of the system. By analyzing their properties, we can gain an understanding of how the whole system works. For example, say I have put together a financial portfolio of different assets. Well, if the risk on these different assets is not related in any way, that is to say, they are all from very different sectors of the economy, well I can calculate the overall risk of my portfolio by simply analyzing the properties of each asset and then summing them up to formulate the total value of the overall portfolio. But what if many of the assets in my portfolio are correlated? If I have acquired many investments within both say the food processing industry and agriculture, well the risk on both is correlated, or assets in both logistic and retail are again interconnected. Because of these correlations, the value of the assets will move together. Thus, the real risk return ratio of my portfolio is no longer defined by that of each asset in isolation, but now by these correlations, that is to say, what is connected to what and in what way are they connected comes to now define the whole system. We often spend a lot of time analyzing individual components and then assume that the whole system is simply an additive function of these parts. But when we turn up the connectivity within a system it is increasingly the relations between components that come to define the overall system, and this is where networks theory finds its relevance as it is all about connectivity.

## Space

Networks have a very different kind of space to the one we are used to. We have spent our lives walking around in a 3-dimensional space, what we called a Euclidean geometry. That is deeply intuitive to us but we need to forget about this, because it is the geometry of things or object, not of connectivity. The geometry of networks is what we call their network topology, and this topology stretches and bends our three-dimensional space around it. For example, say one lives in Istanbul Turkey. If we pull out a map, we will see that Budapest, the capital of Hungary, is much closer than London to Istanbul. But because London is a major hub in the global air transportation system while Budapest in only a minor one, it is much easier to make a connection to London instead of Budapest. So if we put something in this network at Istanbul because of the network's topology, London is essentially closer to it than Budapest. This should give some insight into how a network operates in a different type of geometry than the one we are used to, and thus they cut across our traditional domains, not just in space as in this example, but in all areas making the study of networks a truly interdisciplinary one.

## Emergence

Thirdly, networks represent a very organic type of structure that often emerges from the bottom up, but also has some environmental constraints imposed upon it. Examples of this might be the trading routes that have emerged at different periods in history. During the Middle Ages, traders from Asia would exchange goods with merchants in India and the Middle East, who would, in turn, bring goods to Europe and vice versa; thus emerging an almost global network of trade routes out of the local interactions between merchants. The same can be seen within an ant colony where individual ants leave a trail of pheromones to food sources. Here again, a network emerges from the different ant trails. But none of these connections and the networks that emerge out of them are for free. They cost something to maintain, and thus many of these networks are the product of an interaction between the local elements that are creating the connections and the environment that is placing some resource constraint on its development.

## Nonlinearity

Lastly, complexity and nonlinearity are inherent features of networks. As the number of elements in a network may just grow in a linear fashion as in 1, 2, 3 and so on, the number of connections between them may grow exponentially. So if you take just a small group of people, say 10 or 20, there can be literally billions of different types of networks between them. Nonlinearity is a reoccurring theme with networks, and we can only approach this type of exponential complexity with the use of computers.

When we combine this new way of describing the world that network theory gives us with the powerful tools of computation and a flood of new data sources that we now have available to us, we get a mini-revolution and a new approach called network science that is starting to having a major impact on many areas of science. Networks give us a very intuitive way of visually representing complexity. The model of a network is distinctly visual and this can provide a quick and intuitive overview to a complex system. By just looking at it, we can get a quick sense of how connected it is, what are the key nodes and other critical information to understanding the whole system.

# Network Science

Network science is the area of science and mathematics that tries to analyze and interpret networks based upon data and the use of computer software. Network science is supported by the formal language of graph theory, a relatively new area of mathematics that provides a standardized language with which to talk about and quantify the structure and properties of networks. Presented with a network to analyze, the question turns to what are the features and properties of this network that are of real interest?

## Key Features

The first set of questions we might like to ask relate to individual elements within the network. We want to know what are the nodes within the network, what are the connections between them, and what properties are we really interested in. For example, in a computer network we might not be interested in who owns the different computers and connections, but just interested in the speed of the computers and the bandwidth of the connections. So we need to define what it is about the network we are interested in because as with all models we will be focusing on some information and excluding another.

There is lots of other information we want to know about these individual elements and the connections, such as asking whether they are weighted or not, meaning can we ascribe a value to them. We can talk about a computer network’s bandwidth in megabits per second, but it might not be so easy to do the same with a social network where the relations are of friendship or kinship. We can also ask if these relations go both ways or are just unidirectional. Other questions one might be interested in asking here is how connected is any individual node or how central is it within the overall network.

## Global Properties

The next major set of questions one might ask about a network will relate to its overall structure. Networks are defined by both what happens on the local level, that is how central or connected you are, but also what happens on the global level because the dynamics of the network on the global level feeds back to affect the elements on the local level. Here some of the key questions one might ask about the overall structure of the network include; Firstly, how connected is it? Are there connections between all the parts or are some parts disconnected and separate from others? How dense is this set of connections? If we compare a group of unassociated people waiting at a bus stop with a close-knit group of friends, we will see the density of the network will vary greatly. What are the patterns of clustering within the system? Do we see many small groups or just a few large groups? These are the types of features that will define the overall makeup of the network structure. One key question one might be interested in answering here is if we change some parameter to one of these properties, that is, increase or decrease its value, how will that affect the overall structure to the system?

## Types of Networks

The possible ways in which we can connect even a few elements grows very quickly, and there is of course many different structures to networks possible. It is not possible to create a list of all of these, but what we can do is try and identify some fundamentally different types or models to networks. The first type of network that researchers started to explore was what is called a random network.

By studying randomly generated networks, we get an important insight, which is that most networks are not random. They are created and often defined by the rules under which the elements chose to connect to other elements within the network. Sometimes networks are specifically designed in a top-down fashion, such as the computer network within a corporation, where some systems administrator has specifically designed it in a particular way.

But many of the networks we see around us are not like this. Within many networks such as commercial market, logistics networks, friendship networks, terrorist networks, food webs and so on, the overall network emerges out of local level rules and interactions. When one begins to understand these rules, then one can begin to understand the different types of overall network structures that emerge out of them.

## Network Diffusion

The next set of questions one might want to ask about a network relates to how something will spread out or travel along the network, and this is referred to as diffusion. If we are trying to understand the outbreak of a disease in a given area, we will be trying to understand how it is spreading and what network structures will give rise to rapid or delayed spreading. One would also be interested in how changing a given parameter would affect this. There will be times when diffusion is a positive thing and times when it is not, here we will be talking about how to contain it to prevent disaster spreading. And this would lead naturally to a discussion on network robustness and fragility, how susceptible is the network to failure both from random and strategic attack.

### Network Dynamics

Lastly, how networks change over time presents another set of questions about the network we are analyzing. How does something like the network to a political regime come to form? What are the mechanisms that hold it together? And when does it disintegrate?

# Graph Theory

In the formal language of mathematics, a network is called a graph, and graph theory is the area of mathematics that studies these objects called graphs. Graph theory is a relatively new area of mathematics that gives us a formal language with which to describe networks. The first theory of graphs goes back to 1736. The first textbook came about in 1958 but most of the work within this field is less than a few decades old.

In its essence, a graph is really very simple. It consists of just two parts, what are called vertices and edges. Firstly vertices, a vertex or node is a thing, that is to say, it is an entity, and we can ascribe some value to it. So a person is an example of a node, as is a car, planet, farm, city or molecule. All of these things have static properties that can be quantified, such as the color of our car, the size of our farm, or the weight of our molecule.

Within network science vertices are more often called nodes, so we will be typically using this term during the course. Edges can be defined as a relation of some sort between two or more nodes. This connection may be tangible, as in the cables between computers on a network, or the roads between cities within a national transportation system. Or these edges may by be intangible, such as social relations of friendship. Edges may be also called links, ties or relations. The nodes belonging to an edge are called the ends, endpoints, or end vertices of the edge.

### Graphs

Within graph theory, networks are called graphs, and a graph is defined as a set of edges and a set vertices. A simple graph does not contain loops or multiple edges, but a multi-graph is a graph with multiple edges between nodes. Whereas a simple graph of a transportation system would just tell us if there is a connection between two cities, a multi-graph would show all the different connections between the two cities.

Graphs can be directed or undirected. With an undirected graph, edges have no orientation. For example, a diplomatic relation between two nations may be mutual, and thus have no direction to the edge between the nodes. These undirected graphs have unordered pairs of nodes, that means we can just switch them around. If Jane and Paul are married, we can say Jane is married to Paul, or we can say Paul is married to Jane. It makes no difference, and thus it is an unordered pair.

### Directed Graphs

In contrast to an undirected graph, we have directed graphs, which is a set of nodes connected by edges, where the edges have a direction associated with them. That is typically denoted with arrows indicating the direction. For example, if we were drawing a graph of international trade, the graph might have arrows to indicate the direction to the flow of goods and services. Directed graphs have some order to the relations between nodes, and this can be quite important. A graph is a weighted graph if a number – a weight – is assigned to each edge. These numbers quantify the degree of interaction between the nodes or the volume of exchange. With the trading example above, if we wanted to convert this into a weighted graph, we would then ascribe a quantitative value to the amount of trade between the different nations.

### Multiplex Graphs

So this is the basic language of graphs, but we can extend this language to talk about graphs that have multiple types of edges and nodes, what are called multiplex networks that add a whole new level of complexity to our representation, allowing us to capture how different networks interrelate and overlap to affect each other.

# Degree of Connectivity

The degree of connectivity to a node in a network is a measure of the number of in and out links the node has to other nodes. This degree of connectivity can be interpreted in terms of the immediate likelihood of a node catching whatever is flowing through the network. For example, the higher one's degree of connectivity within a given social network the more likely you are to hear about some piece of news that is being shared, because you have many more channels through which to intercept it. Of course, this works both ways as it might not be news that is spreading on the network, but instead a virus.

## Unweighted Network

If we are analyzing an undirected and unweighted network, a node's degree of connectivity will then simply be a summation of all the links that the node has. If the graph is directed then we can refine our analysis by dividing this into a measure of the amount of in and out links. With an example of nations trading, the in-degree of any node would be the total number of import relations it has with other nations and the out-degree would be the total number of exporting relations it has. If this is a weighted graph, we can then, of course, refine our model further by placing quantitative values on each edge.

## Adjacency Matrix

If an edge exists between node A and B, then we say they are adjacent. So if we take a map of a subway, we could say that each station or node is adjacent to any other station that is just one stop away from it. We can then capture all the relations within a network by creating an adjacency matrix. We can create a simple 2 by 2 matrix to capture the connections within a network by placing a 1 at the cross section between two nodes if they are adjacent and a 0 if not, with the end result being a table of all the connections within the system; this is how a graph is represented in computer code.

## Path & Geodesic

Another thing we might be interested in is looking at how two nodes in a network are connected, that is to say, the channel or paths through a network from one node to another, and this is called a walk. A walk on a graph is a sequence of adjacent vertices where repetition is allowed. With a walk, we are simply going from one node to the next along a sequence of edges. A path is a walk without revisiting any nodes, that is a sequence of links from the first node to the last without repetition.

As an example of why this might be of interest to us, we could cite the so-called traveling salesman problem, which involves a salesman that has to visit a number of different cities within a particular region. He, of course, wants to avoid walks where he will be retracing the same ground and try and find a path that will not involve any repetition of the cities he has to visit. The traveling salesman will also be interested in finding the shortest path between each place he has to visit. This shortest path between two nodes on a graph is called the geodesic, and it represents the fewest number of links we need to traverse in order to get from any given node to another.

# Network Connectivity

One of the defining features of a network is its overall degree of connectivity, which might qualify as *the* defining feature. Going from a system with a low degree of connectivity to one with a high degree of connectivity is not just a quantitative change in the number of edges within the network. It is also a qualitative change. It marks a shift from a component based regime, where we need to firstly think about the components of the system, their properties in isolation, and their linear interactions; to a relational based regime where we need to first model how the system is interconnected.

One way of contextualizing the degree of connectivity to a network is by talking about how easy or difficult it is for a node in the network to make a connection with another because the overall connectivity emerges out of the local actions of the nodes in the network. If we make it difficult for them to interact then there will be a lower overall connectivity.

## Example

If we take any network, say a logistics network, we can ask under what circumstances are the nodes more likely to interact. In this case, the nodes are producers, distributors, and consumers, and they will be more likely to interact as the cost of transportation and trade restrictions are reduced. The development of the global economy over the past few decades could be cited as an example here.

Through the reduction in trade tariffs and advancements in transportation and communications, the ease of interaction between producers and distributors on a global level has increased, resulting in the increased density of logistics and trade networks as regional and national economies have become integrated into the global economy. Just to put some real figures to this, in America the logistic cost of transporting some freight is thought to be around 5% of the cost of the goods, whereas in China it is thought to be around 12% to 13%. You can then imagine how a business will factor this into their choices as to whether they supply to far off distributors or not, and thus whether the network becomes denser or more sparse.

## Coupling Parameter

One way of quantifying this concept of the overall connectivity to a network is with reference to its density. The density of a network is defined as a ratio of the number of edges to the number of possible edges, and this will also correlate to the average degree of connectivity to the nodes in the network. So when we increase our coupling parameter, we are increasing the density of the network and the average degree of connectivity.

This coupling parameter to a system can, of course, be defined by many different things. Depending on the network we are dealing with, it may be economic as in our example above which was measured in terms of the financial cost of transportation. Or it may be measuring the climatic condition to an ecosystem, where we could quantify it in terms of the average environmental temperature. As we reduce the temperature, the number and density of interactions between creatures reduce as they hibernate in isolation. We could also think about the formality of a social setting as a parameter. As we reduce the formality of the setting, say by having an office party, people’s social inhibitions are reduced and they are more likely to interact.

As we turn this coupling parameter up or down, thus requiring the nodes to exert more or fewer resources in order to create a connection, we would expect the level of integration within the network to increase or decrease. The easier it is for elements to create a connection the more connections and the longer these connections can be, thus working to integrate the entire system. And inversely, the more resistance there is for nodes to create connections the less there will be and the network will disintegrate, with the most costly ones, that is those that are maintained over a greater distance, being the first to go. But this does not always change in a linear fashion.

## Nonlinear Growth

The amount of connectivity within the network will be primarily defined by how much resources a node has to expend in order to make that connection. The amount of resources that a node will have to expend in order to create a connection will grow in a somewhat proportional fashion to the length of the relations. So if I am walking to the local shop, I will have to exert a certain amount of energy to do this. And if the shop is twice as far away, I will have to expend twice as much resources in order to create that connection. This is a linear progression. But this simple linear scaling is not always the case. Say I am taking an intercontinental flight, a journey of some 2000 kilometers. This does not require 10 times more effort on my behalf than a local flight of 200 kilometers. Because of this, the degree of connectivity in the system may not always grow in a simple linear fashion, but because of this nonlinearity the level of connectivity can grow or decay in an exponential and rapid fashion resulting in there being tipping points and phase transitions.

# Network Centrality

Centrality is a measure that tells us how influential or significant a node is within the overall network.[1] This concept of significance will have different meanings depending on the type of network we are analyzing. In some ways, centrality indices are answers to the question “What characterizes an important node?” From this measurement of centrality, we can get some idea of the node’s position within the overall network. The degree of a node’s connectivity is probably the simplest and most basic measure of centrality.

We can measure the degree of a node by looking at the number of other nodes it is connected to vs. the total it could possibly be connected to. But this measurement of degree only really captures what is happening locally around that node. It does not really tell us where the node lies in the network, which is needed to get a proper understanding of its degree centrality and overall influence.

## Key Parameters

The concept of centrality is quite a bit more complex than that of its degree of connectivity and may often depend on the context, but we will present some of the most important parameters for trying to capture the significance of any given node within a network. The significance of a node can be thought of in two ways, firstly how much of the networks resources flow through the node and secondly how critical is the node to that flow, i.e. can it be replaced? So a bridge within a nation’s transportation network may be very significant because it carries a very large percentage of the traffic or because it is the only bridge between two important locations.

The four most significant metrics for quantifying this are; Firstly, a node’s degree of connectivity, which is a primary metric that defined its degree of significance within its local environment. Secondly, we have what are called closeness centrality measures that try to capture how close a node is to any other node in the network, that is, how quickly or easily can the node reach other nodes. Betweenness is a third metric we might use, which is trying to capture the node’s role as a connector or bridge between other groups of nodes. Lastly, we have prestige measures that are trying to describe how significant you are based upon how significant the nodes you are connected to are. Again, which one of these works best will be context dependent.

## Closeness

Closeness may be defined as the reciprocal of farness, where the farness of a given node is defined as the sum of its distances to all other nodes. Thus, the more central a node is the lower its total distance to all other nodes. Closeness can be regarded as a measure of how long it will take to spread something, such as information, from the node of interest to all other nodes sequentially. We can understand how this correlates to the node's significance in that it is a measurement of the node's capacity to affect all the other elements in the network.

## Betweenness

Betweenness, as mentioned, is really talking about how critical a node is to a network in its functioning as a unique bridging point between other nodes in the network. Betweenness centrality quantifies the number of times a node acts as a bridge along the shortest path between two other nodes. In this formulation, vertices that have a high probability of occurring on a randomly chosen shortest path between two vertices have a high betweenness value.

## Eigenvector centrality

Our last measure is trying to capture how connected the nodes that a given node is connected to are. So instead of looking at the total amount of connections you have, it is more interested in the value of those connections. One way of capturing this is called Eigenvector centrality. Eigenvector centrality assigns relative scores to all nodes in the network based on the concept that connections to highly connected nodes contribute more than connections to nodes with lower degrees of connectivity. Eigenvector centrality is one measure used by web search engines to try and rank the relative importance of a website by looking at the importance of the websites that link to it.

# Network Topology

Network topology refers to the overall makeup and structure to a network, with the word topology meaning the way in which constituent parts are interrelated or arranged. Within the context of network theory, it defines the way different nodes are placed and interconnected with respect to each other and the overall patterns that emerge out of this.

## Local Interactions

The overall topology of a network is often a product of the local interactions between the nodes. For example, someone builds a protocol for two computers to exchange information over a network and shares it with a colleague. Other people see the utility of it and connect to this little network, and then more people as the network grows, until 25 years later we have a massive network of networks that is the internet. No one planned the internet just as one really planned the global financial networks that have emerged over the past few decades. Traders, investors, and institutions set up connections wherever they thought there was a viable return on investment. But now that these networks are here, their overall markup feeds back to affect us the users. Networks may start out quite random but they often develop into some stable overall structure, and understanding the patterns to this overall structure is of central importance in network theory.

## Key Factors

Some of the most important factors influencing the overall topology to a network include; its degree of connectivity, the number of nodes in the network, and its overall pattern of connectivity, i.e. how centralized or distributed it is. With connectivity, we are really talking about the density of the connections in the system. The degree of connectivity will largely be a product of how easy or difficult it is for any two nodes to form a connection. As we reduce the barriers to interaction, we will see the network become denser, integrated and increasingly defined by the structure and make-up of the network, as opposed to the components in isolation.

Another macro scale property to the network that is of importance with respect to its overall topology is its size. By size, we simply mean the number of nodes. This may sound like a trivial factor but scale can matter, as sometimes more is not just more –

it can, in fact, be different. Think about living in a small rural community where everyone knows everyone by just one or two degrees of separation vs. living in a large urban metropolis where the anonymity of much longer path lengths between people creates new types of social dynamics. Lastly, the network’s overall pattern of connectedness is an important factor.

The way in which a network is connected plays a large part in how networks are analyzed and interpreted. Due to some common set of properties shared by a subset of the system, we often get subsystems forming within networks. These subsystems are called clusters and often have a significant effect on the networks makeup. For example, we might think here about the clustering in the different cultural groups around the world. Although two cultures like that of France and Italy are different, they share a common Greco-Roman heritage that gives them and other European countries a set of common features through which they form a cultural cluster within the network of global cultures.

# Network Diameter

The size of a network is important not so much because of the sheer quantity of elements we are dealing with, but more because it sets the context for how close or far on average one node in the network is from another. And this is important because it will tell us how quickly something will spread through the network and also how integrated different components in the network are likely to be.

Making a connection within a network, or traveling from one node to another is rarely free. It typically costs some resource, whether this is the cost of fuel to travel in a transportation network, the laying of cables to transport information, or the risk of rejection when you ask someone out on a date. The further we have to travel along a network to get from A to B the more it will cost and the less likely it will occur, with the result being a lower level of integration to the system.

## Example

As an example of this, we might think of the Russian empire in the 1900s, a vast landmass spanning from Europe to East Asia. But without any coherent transportation network connecting it, it was continuously under threat of falling to pieces. With the building of the Trans-Siberian rail network, information, goods and resources could eventually diffuse to the different parts of the system, and it is this interaction between components to the system that gives it cohesion and integration. We may be dealing with a very large system, but if there is no diffusion or interaction across the network it ceases to function as an entirety.

Two important metrics for capturing this overall distance between nodes are; the network's diameter and its average path length. The diameter of a network is simply the longest of all the geodesics in the network, where geodesic means the shortest path between two nodes. When we are asking for the diameter of a network we are looking at all the shortest paths and then choosing the longest one, and this will give us an idea of how far something might have to travel to get all the way across the network. The average path length is calculated by finding the shortest path between all pairs of nodes, adding them up, and then dividing by the total number of pairs. This will show us the number of steps on average it takes to get from one member of the network to another.

## Small World Phenomenon

We can take a real world network and ask what is its average path length. For example, a number of years ago researchers studied the social network of Facebook when it had approximately 721 million users with 69 billion friendship connections between them. The average path length turned out to be just 4.74 intermediary connections.[2] This appears to be an extraordinary low distance between any two members of such a large network. Quoting one of Facebook's spokesperson on this finding, "In these two works, we show how the Facebook social network is at once both global and local. It connects people who are far apart, but also has the dense local structure we see in small communities." This is what is called the small-world phenomena. We can see from this that the question of how close things are in a network is not just a product of its size, but it is also a product of the structure to the network as we would expect.

## Topology

Even when we have a very larger network, if it has the right structure then we will be able to reach more nodes in a shorter path length than if the network was smaller but had a structure that gave it only a linear relation between distance traveled and nodes connected to. We often think about and measure size and scale in terms of some static quantity, the number of people in a city or the creatures in an ecosystem. But as networks are all about connectivity, what really matters with respect to scale here is how far you are away from other nodes which can be dramatically altered by simply restructuring the network, and thus scale becomes much more subjective and relative to the topology of the network.

# Network Degree Distribution

A key parameter to the makeup of any network is the network's degree distribution. Degree distribution tries to capture the difference in the degree of connectivity between nodes in a graph. It is really asking the question, do all the nodes have roughly the same amount of connections or do some have very many while others have very few connections? By answering this, we will get an idea of how centralized or distributed it is, which is a defining factor to networks telling us how something will flow through it, which nodes have influence, or how quickly can we affect the entire network.

## Degree Distribution Spectrum

It is possible to define a spectrum for the network's degree of distribution starting from systems with very homogeneous degree distribution, that is, all nodes have a relatively similar amount of connections. Here we will be talking about random networks and distributed systems where we have a relatively even topology to the network, but as we turn our degree distribution parameter up we will start to see hubs appearing. These types of networks are described as decentralized, implying that unlike our distributed graph where there was no real center, these have a number of different central hubs to them.

These decentralized networks have the small-world property that we mentioned earlier, making them very effective at connecting a large amount of elements within a short average path length. Lastly, if we turn up our degree distribution parameter to make a very large disparity between the node's different degrees of connectivity, we will start to get centralized networks with one or few dominant nodes and many nodes with a relatively low level of connectivity. This type of network is captured within a model called a scale-free or power law network that we will be talking about in a later section.

We might then ask why we get these different networks with fundamentally different degree distributions? If we start out with a random network, we will be able to see that most networks are in fact not random at all. Birds do not just choose at random what other creatures they are going to prey on within a food web. People do not randomly choose their friends and transport authorities do not just randomly lay down highways between any two locations. These connections are of course made under specific rules that govern why and to which other nodes any node will make a connection with. And it is out of the aggregate behavior of these nodes interacting that we get networks that have specific and widely encountered properties, meaning we do not live in a world of random networks but in fact a world of networks that have a specific structure that has emerged out of these local rules.

### Importance

Degree distribution parameter may just be a quantitative parameter, but changing it can have a qualitative effect on the network we are dealing with. To illustrate this, we might take an example of a political network. Political social networks span from the highly centralized form of dictatorship, where all sociopolitical connections lead back to one dominant node, to at the other end of the extreme some kind of egalitarianism where responsibility and authority are distributed out in a pluralistic fashion. These different network structures will have a systemic effect and influence on almost all areas of the social and cultural fabric.

# Network Clustering

The way in which a network is connected plays a large part in how we analyze and interpret it. When analyzing connectedness and clustering, we are asking how integrated or fractured the overall network system is, how these different major sub-systems are distributed out and their local characteristics. A graph can be said to be connected if for any node in the graph there is a path to any other node. When the graph is not connected then there will be a number of what we call components to it. A component is a subset of nodes and edges within a graph that are fully connected. Thus, for a node to be part of a component it must be connected to all the other nodes in that component.
A cluster is simply a subset of the nodes and edges in a graph that possess certain common characteristics or relate to each other in a particular way, forming some domain-specific structure. Whereas a component is simply referring to whether a given set of nodes are all connected or not, a cluster is referring to how they are connected and how much they are connected, that is the frequency of links between

a given subset of nodes.

In order to model the degree of clustering of a subset of nodes, we simply take a node and look at how connected a node it links to is to other nodes that it is also connected to. For example, if this was a social network of friends, we would be asking how many of your friends know your other friends. The more your friends are interconnected the more clustered the subset is said to be. This clustering within social networks is also called a clique. A clique is a group of people who interact with each other more regularly and intensely than others in the same setting.

## Homophily

Within this social context, clustering can be correlated to homophily, where homophily describes the phenomenon where people tend to form connections with those similar to themselves, as captured in the famous saying "birds of a feather flock together." We might think of clustering coming from the fact that the interaction between nodes with similar attributes will often require fewer resources than interactions between nodes with different attributes. For example, between two cultures there may be a language barrier, or between different devices on a network that might have different protocols. Equally, clustering may be due to physical constraints of the resource expenditure required to maintain them over a greater distance, thus resulting in a clustering around a geographic neighborhood.

## Topology

Understanding the different local conditions that have created clustering within a network are important for understanding why the network is distributed out into the topology that it has, how you can work to integrate it or disintegrate it, and how something will propagate across the network, as each one of these clusters will have its own unique set of properties within the whole, making it particularly receptive or resistant to a given phenomenon. For example, we might be analyzing a political network, with each cluster in this network representing a different set of ideologies, social values, and policy agendas that are receptive to different messages.

Or as another example, by understanding that different clustering groups on a computer network may represent different operating systems, we will be able to better understand why a virus has rapidly spread in one part of the network but not in another. And also by understanding this local clustering condition, we will be able to better approach integrating them into the broader network. The clustering coefficient of a node is then a method for measuring the degree of a local cluster. There are a number of such methods for measuring this but they are essentially trying to capture the ratio of existing links connecting a node's neighbors to each other relative to the maximum possible number of such links that could exist between them. A high clustering coefficient for a network is an indication of the small-world phenomenon.

# Random & Distributed Networks

A random network is more formally termed the Erdős–Rényi random graph model, so named after two mathematicians who first introduced a set of models for random graphs in the mid 20th century. As the name implies, this type of network is generated by simply taking a set of nodes and randomly placing links between them with some given probability. So we just take two nodes in the network and we roll a dice to see if there will be a connection between them or not. The higher we set our probability the more likely there will be a connection, and thus the more connected the overall graph will become. This may be defined as a simple system in that once one has decided how many nodes there will be, it is then really just defined by a single parameter, that is the probability parameter for the likelihood that any two nodes will form a connection.

## Normal Distribution

Looking at the degree distribution of this network would reveal that it follows a normal distribution. Because it was randomly generated, there will be some difference in the distribution of degrees of connectivity among the nodes. Some will have one degree, some five, but there will be a well-defined normal or average degree. In this distribution, there will be very few nodes with a very large degree, and very few with a very low degree. Most will tend towards the normal amount of connections.

Unlike real world networks, there is low clustering in random networks. Therefore, the resulting network very rarely contains highly connected nodes. Consequently, a random network is not a good candidate model for the highly connected architecture that characterizes many of the networks we see around us. Although a useful theoretical exercise, random networks, in general, do not represent networks in the real-world. They are considered far more random because real-world networks are typically created to serve some function and are constrained by some limiting resource that gives them a more distinct pattern.

If we look at some network like the traditional trade routes across the Sahara desert in Africa, it may look somewhat random at first glance, but we know that it is not because of the caravans of camels and traders who created these networks. Setting out to cross the Sahara in any random direction would have of course been fatal to them.

## Factors in Network Structure

This helps illustrate the two key factors to generating any given structure to a network. Firstly, the context or environmental constraints the network is under. Different types of networks are all under different types of environmental constraints. For example, this may be the geological constraints placed upon the travelers in our example above. It may be the physiological constraints placed upon the metabolic network with the human body, or it may be the financial constraints placed upon a logistic network.

All of these represent resistance to network formation that the environment places on the network, but inversely we can look at this the other way around, asking what methods the nodes in the network use to overcome these constraints. Travelers in the Sahara would use prior knowledge encoded in maps as to where the water wells were in order to overcome the arid environmental constraint placed upon them.

A national airline because of the limitations on finance may not be able to run a direct route between every city within the country, but will get around this by creating a hub and spoke network so as to reach all locations. Again, this is a method or strategy for overcoming resource constraints. And out of the interaction between these environmental constraints and the methods used by the nodes to overcome them, we will get a particular overall structure to the network that makes them distinctly different from our random network.

## Distributed Networks

A distributed network is in many ways quite similar to a random model. It is defined by a low degree distribution level, meaning all or most of the nodes have the same degree of connectivity. And as there are no dominant nodes to provide global functions for the entire network, each node must contribute equally to the network's maintenance.[3] And as there is no real global coordination in a distributed network, nodes can have a very high degree of autonomy, as they are largely self-sufficient and independent from nodes outside of their neighborhood.

An example of a distributed network might be a community alert group, where each member of the community has equal responsibility and authority to act when there is an event that others should know about. There is no hierarchy, and in this example, the network is only actualized when needed, thus placing very limited constraints on its members. Within the world of computing distributed networks are also called mesh networks.

### Advantages & Disadvantage

Distributed networks have a number of advantages and disadvantage. On the positive side, they may be very robust to failure. As there is no critical or strategic nodes in the network, any node can theoretically be replaced by any other, and as we have already noted elements may have a high degree of autonomy, with little network maintenance tax placed upon them. But also this type of network can be less efficient in many circumstances. Without centralized nodes, there cannot be any centralized batch processing that leverages economies of scale, and diffusion across the network can be slow as there are no central hubs with which to reach many nodes in a single hop. This can also create problems in terms of coordinating the network as a whole. In many ways, a distributed network represents a system in a fine balance and relatively stable state and this is often not what we see when we look at real-world networks.

# Centralized & Power Law Networks

Centralized networks represent networks with a very high degree distribution, meaning in this type of network structure there will be very many nodes with a very low level of connectivity and a very few, or maybe just one node with an exceptionally high degree of connectivity. Thus, they are very heterogeneous and unequal in terms of how connected and influential the different nodes in the network are.

A good example of a centralized network might be global banking activity with nodes representing the absolute size of assets booked in the respective jurisdiction and the edges between them the exchange of financial assets. A very few core nodes dominate the global financial network, with approximately 200 countries in the world, but the 19 largest jurisdictions in terms of capital together are responsible for over 90% of the assets. This type of centralized structure to a network is surprisingly prevalent in our world and there are many other examples of it, such as social networks where a very few people may have millions of people connected to them and the vast majority very few.

## Power Laws

These highly centralized networks are more formally called scale-free or power law networks, that describe a power or exponential relationship between the degree of connectivity a node has and the frequency of its occurrence. These power law networks are really defined by the mathematics that is behind them. The number of nodes with degree x is proportional to 1 over x squared. For example, the number of nodes with degree 2 is one fourth of all the nodes. The number of nodes with degree 3 is one-ninth of the nodes. The number of nodes with degree 10 is proportional to one hundredth. Power law distribution like this creates what is called a long tail distribution. The long tail means there can be nodes with a very high degree but there will also be very many with a very low degree of connectivity giving us our centralized network.

This type of power law graph was first discovered within the degree distribution of websites on the internet, with some websites like Google and Yahoo having very many links into them, but there also being very many sites out on the web that have very few links into them. Since then it has been discovered in many types of very different networks, such as in metabolic networks where the essential molecules of ATP and ADP that provide the energy to fuel cells play a central role interacting with very many different molecules, whereas most of the molecules interact with very few others, thus making these two molecules hubs in the metabolic networks fueling the cells in our bodies.

## Preferential Attachment

This power law has also been documented in the frequency of citations between academic papers and within the social network between of Hollywood actors. This scale-free property to networks is then interesting because it appears regularly and across all forms of networks, from the Internet to social groups to biological systems. The power law distribution to a network like the World Wide Web is often explained with reference to what is called preferential attachment. Preferential attachment describes how a resource is distributed among a number of nodes according to how much they already have so that those who already have a lot receive more than those who have little. In more familiar terms this is called “the rich get richer.”[4]

Within this model, if you are say, building a website and choosing which other websites to link to, then you will be twice as likely to link to a website that has twice as many links as another. So to formalize that a bit better, the probability that you will make a link to a site is proportional to the size of the site.

If a network was created under these rules then we should get a power law distribution, but in reality, this is quite a simplified mode. It should just give you an idea for some of the mechanics behind these power law networks. Why we have these very large centralized nodes in the financial system is of course much more complex than this, involving a number of different parameters. Most notable among these is the actual quality of the service that the node is providing, not just its size.

### Robustness

With respect to their robustness, centralized networks can be very robust or very fragile depending on if we remove nodes randomly or strategically. If we were removing nodes randomly, they will be very robust to failure because the vast majority of the nodes have a very low degree of connectivity, and thus we would likely be removing one of these insignificant nodes with little effect on the overall network. But inversely, if we were to remove a node strategically, that is to say, purposefully choosing the node we remove in order to maximize the damage we are doing, then these centralized networks are very susceptible to failure of this kind because we just have to remove one of the giant hubs that are critical in their role, connecting many smaller hubs and the system will be affected greatly by this.

# Decentralized & Small World Networks

A decentralized network is one without any overall dominant central hub. But instead, a network that has some nodes with a higher degree of connectivity than others, giving the overall topology local clusters with local hubs. We call this network model that has local hubs but still relatively little overall differentiation to it a decentralized network. There may be some overall center to it but it is still defined largely by what is happening on the local or regional level.

To take an example of a decentralized network, we could cite the urban network of contemporary Germany. Unlike other countries such as Japan or Nigeria whose urban network is dominated by a primary node, Germany’s urban infrastructure, and the services that it provides are distributed out into a number of important centers. For example, the primary air transportation and financial hub is in Frankford, the political capital in Berlin, with Munich having the strongest economy. Each of maybe five or ten centers play a very important role in maintaining the network. There are of course many more examples we could cite, such as conglomerate corporations, political federations or distributed computer networks.

So why do we get these local level hub and spoke structures emerging? There are a number of reasons for this, but many tie back to the fact that the system is under certain environmental resource constraints, and it will only be possible for nodes to overcome some of these constraints by combining their resources. This, coupled with batch processing and the economies of scale that it enables are behind the formation of many hubs, from banks that amass financial resources to be able to fund large projects, to international airports, to the emergence of factories as local hubs in manufacturing networks. These hubs then serve the function of connecting nodes locally, but also connecting them globally to other hubs in the network. The result then is local clustering but also some global connections between clusters, and this gives us the small-world phenomena previously mentioned.

## Small World

A small-world network is a type of graph in which most nodes are not neighbors of one another, but most nodes can be reached from any other by a small number of connections. A certain category of small-world network was identified by Duncan Watts and Steven Strogatz in 1998.[2]Watts and Strogatz measured that in fact many real-world networks have a small average shortest path length, and also a clustering coefficient significantly higher than expected by random chance. This is in many ways quite counter-intuitive, as what it is saying is that even though there is a significant amount of clustering in these networks, meaning that nodes are typically highly connected to other local nodes in their cluster, and if we had quite a large network like this, then we would expect there to be quite a long shortest path length, which is not the case here.

Eventually they came up with a model that captured this phenomenon. It involved starting with a ring lattice where all the nodes are only locally connected, and thus has high clustering, but then randomly picking some links to rewire so that they would likely not connect to their local cluster but somewhere else in the network. They found that you do not need to randomly rewire very many links before the shortest diameter starts dropping very quickly, and from this we are able to capture the small-world phenomena.

## Six Degrees of Separation

The small-world phenomena has since gone on to be popularized in the six degrees of separation hypothesis, which is a theory that everyone is just six or fewer steps away from any other person in the world, so that a chain of “a friend of a friend” connections can be made between any two people in a maximum of six steps. Many empirical graphs are well-modeled by small-world networks. Social networks, website links on Internet, wikis such as Wikipedia, and gene networks all exhibit this small-world characteristic.

We can see the small-world phenomena behind our decentralized network model, as it had these local clusters with hubs, with the hubs making global connections allowing for a relatively efficient set of overall connections to the system, without having to expend t0o much resources on maintaining very many global connections that are likely to be expensive and difficult to maintain.

# Network Dynamics

The study of how networks form, grow and change over time is a relatively new area of research, but it is critical to understand how to foster the development of some types of networks and reduce the development of others. For example, researchers have studied innovation as a process of diffusion across a network. Traditionally, research in graph theory focuses its attention very much on studying graphs that are static.

However, almost all real networks are dynamic in nature and how they have evolved and changed over time is a defining feature to their topology and properties. As network theory is a very new subject, much of it is still focused on trying to explore the basics of static graphs, as the study of their dynamics results in the addition of a whole new set of parameters to our models and takes us to a new level of complexity; much of which remains unexplored.

The development of a network may involve adding more nodes to it, but also more interestingly, adding links to it that increase the overall connectivity. In random network models, links are just placed between nodes at random with some given probability. “Growing” the network here just meant increasing this probability so as to have more links develop over time. One interesting thing we find when we do this is that there are thresholds and phase transitions during the network’s development. By thresholds we simply mean that by gradually increasing the link probability parameter, some property to the network suddenly appears when we pass a critical value.

## Thresholds

The first threshold is when the average degree goes above 1, over the total number of nodes in the network, as at this threshold, we start to get our first connection. At degree one, that is, when every node has on average one connection, the network starts to appear connected. We see one giant component emerging within the network, i.e. one dominant cluster, and we start to have cycles, which means there are feedback loops in the network. Another threshold occurs when nodes have an average degree of log(n) at this point everything starts to be connected, meaning there is typically a path to all other nodes in the network. This is what we see in random networks, but most real-world networks are not random as they are subject to some resource constraints and they have preferential attachment, giving them clusters that we do not see in these random graphs.

## Network Percolation

One way of thinking about how real-world networks form is through the lens of percolation theory. Percolation theory looks at how something filters or percolates through something else, like a liquid filtering through some mesh structure in a material. Or we might think about some water running down the side of a hill. As it does, the water will find the path of least resistance, creating channels and furrows on the side of the hill. This network formation is then the product of the resource constraints that its environment placed upon it, but the constraints are unevenly distributed and the network's topology is then reflecting this as it follows the paths of least resistance avoiding the toughest material.

In order to demonstrate the general relevance of this, we will take some other examples. If say, we put on cheap flights from one city to another then people will start using this transportation link because of financial constraints. Or because of the phenomenon of homophily within social networks, we will get the same percolation dynamics where it will be easier for people to make links with people who are similar to themselves than with others; again creating a particular structure based on the social constraints within the system.

To incorporate this into our model, we would need to add in the attributes or properties of the nodes within the network where they will form links depending upon these properties. In a social network, these attributes might be age, gender, income etc. and people will have a preferential attachment. We would then create a probability for the likelihood that any node will connect with another of the same kind compared to whether it will make a connection with a node of a different kind.

The result of including these factors into our representation would be a much more realistic model where we see local clustering and some distant relations. Behind this relatively abstract model to network development presented here is a much more complex set of questions about the local incentives of the nodes in the network. Another way of looking at network formation within this model is from the perspective of the nodes and the local rules they are operating under. To do this, we might use game theory that looks at the incentives of the individuals within the network and their payoff for forming a connection or breaking one.

## Network Effect

Another key driver behind the formation of real-world networks is the so-called network effect and Metcalfe's law, which states that the value of a network grows as a square of the number of the nodes in the network.  The network effect arises when users derive value from the presence of other users, with the telephone being a classical example. The more people that join the network the more valuable it is. Thus, there is a positive feedback loop where more people joining feeds back to make the network more valuable. That in turn draws more people and so on. In this way we can get the exponential growth that we have seen in the rise of many network organizations like Twitter and Facebook.

We might note the development of these networks is nonlinear with distinct tipping points, because it requires a critical mass of users for something like a computer operating system to have any value. But once you have gone beyond that critical mass it then becomes very valuable because you are able to now interoperate with all these other users. This is why free is a good marketing strategy for some I.T. startups, because it is all about getting this critical mass. Once you have it then the network effect kicks-in and it become a “must have.”

## Network Formation

Most real-world networks are not random, during their formation, they were subjected to certain environmental and resource constraints that shaped their formation as they developed in a particular non-random fashion. Added to this, most economic networks are user-generated. They have been formed out of local nodes choosing to make connections. Thus, both the local rules under which agents are making these connections and the environmental constraints they are under are both defining factors in the network's formation. For example, if we take a trade network, we need to know what are the physical constraints and the socio-political constraints that are inhibiting the formation of the network, and inversely, what are the set of rules under which agents are choosing to make connections.

## Nonlinearity

The growth of a network may be nonlinear, meaning there will likely be sub-linear growth up to a certain tipping point, and then positive feedback will kick-in to give us super-linear exponential growth. In this way, something like the Internet can lay relatively dormant for a long time and then take off rapidly. An important thing to recognize in the growth of a network is the fact that whereas the number of nodes in the network may grow in a linear fashion, as in one, two, three etc., the number of edges can grow in a super-linear fashion. With 1 node, we have 0 links. With 2 nodes, we can have 1 link. With 3, we can have 3 links. With 4 nodes, we can now have 6 links. With 5, we can have 10. With 6 nodes, we can have 15 possible links.

Whereas the number of edges started off lower than the number of nodes, it will likely sooner or later catch up with it and then outgrow it rapidly. In this example, the number of edges caught up with the number of nodes very quickly because we were talking about the maximum possible number of links, but typically in reality, not every node will be fully connected, and thus it may often take a lot longer for it to catch up. But once it does, we will start to move from a component-based regime to a relational regime and the connections will add significant value to the system. We will get a positive feedback loop and the system may then grow exponentially. This is called the network effect.

For example, the network effect can be seen in stock markets and derivatives exchanges. Market liquidity is a major determinant of transaction cost in the sale or purchase of a security. As the number of buyers and sellers on an exchange increases, liquidity increases, and transaction costs decrease. This then attracts a larger number of buyers and sellers to the exchange. Thus, we get a positive feedback loop that is behind the network effect. In order to understand this process of network development better we will go over each stage individually.

## Sub-Linear

In the initial phase of a network’s formation, due to the limited number of nodes and connections in the network, the value of joining that network may, in fact, be negative because of the opportunity cost. Joining this network may well exclude you from joining another more mature network that already has a lot of network value. For example, if you choose to adopt a Linux operating system, you will be limiting your capacity to interoperate with over one billion users of Windows. Thus, in terms of opportunity cost, you are actually having to pay to be part of this burgeoning Linux network, and the same would be true for a social network, digital currencies and many other types of networks that have not reached a critical mass.

These early adopters are typically special interest users that particularly care about this service and are prepared to pay the opportunity cost. It is these early enthusiasts that really matter because with them your network may be able to reach the critical mass; without them, you will not. And reaching this critical mass beyond which the network effect will take hold is the key factor in the early formation of the network.

A key parameter here is how much of the value of using this system is in the components vs. the connections. If there is value inherent to the product without connecting it, such as would be the case for a washing machine, then early adoption is not very difficult. But other things are very much dependent upon their connections such as the telephone, where it will be very difficult to get the original users because there is no value in the system without the existence of others to connect to. The role of expectations is very important here as if people do not expect the network to grow they will not join and it will not reach the critical mass. If their expectations are positive then it may well reach this threshold.

## Tipping Points

If enough nodes join the network, then we may reach the critical mass and get a tipping point. The tipping point is the critical point in the system's development as it defines where positive feedback will gain traction leading to rapid and irreversible state change.

The term critical mass is said to have originated in the field of epidemiology when the spreading of an infectious disease reaches a point beyond any local ability to control it from spreading more widely. It is in many ways analogy to a phase transition. Marketers use the term to denote a threshold that, once reached, will result in additional sales. At the point of critical mass, the value obtained from the good or service is greater than or equal to the price paid for it. Beyond this, it becomes much more attractive for people to join as the value is continually going up as each new user joining creates a high surplus value for the next prospective user.

With this positive feedback loop, we can get the bandwagon effect where agents couple to the network without any intrinsic evaluation for, or knowledge of the actual phenomena, but simply joint to gain the benefit of the network effect in the way that someone might adopt a certain ideology for fashion without knowledge of it, simply to be socially accepted.

## Peek

The bandwagon effect can lead to overcapacity, as the increasing number of users generally cannot continue indefinitely. After a certain point, many networks become either congested or saturated, stopping future uptake. Congestion occurs due to overuse. As an example, we might think about the telephone network. While the number of users is below the congestion point, each additional user adds additional value to every other customer. However, at some point, the addition of an extra user exceeds the capacity of the existing system. After this point, each additional user decreases the value obtained by every other user.

If this is the case, then the next critical point is where the value obtained goes back down to where it approximates the price paid. The network will cease to grow at this point, and the system must be enlarged to enable future growth. This is the case for centralized systems, but may not be the case for distributed networks. New peer-to-peer network models such as Bitcoin may always defy congestion. True peer-to-peer networks are designed to distribute out the network's load amongst their users. This theoretically allows peer-to-peer networks to scale somewhat indefinitely, at least until market saturation.

## Negative Externality

But there is also a flip side to the network effect and network development, which is crowding out and the lock-in. Due to the importance of interoperability within network economies, there is a strong attractor toward everything converging onto the same network, the same set of standards or protocols resulting in lock-in.

Network effects are notorious for causing lock- in with the most-cited examples being Microsoft products and the QWERTY keyboard. In the previous example where the network effect created liquid markets, it is also apparent in the difficulty that startup exchanges have in dislodging a dominant exchange. For example, the Chicago Board of Trade has retained overwhelming dominance of trading in US Treasury bond futures despite the startup of Eurex US trading of identical futures contracts. Mitigating these negative externalities mean maintaining an open vendor-neutral network within which new standards and protocols can be incorporated. The success of the Internet is in many ways in its openness, net neutrality and the fact that no one owns it.

## Diffusion

The rules under which a network was created and developed will play a large role in how something will spread across it and ultimately how robust it is to failure. The first thing to note with respect to network diffusion and robustness is that connectivity can both add and reduce to the system's robustness. It works both ways. Connectivity is important for integrating the system and it is this integration that gives the system its overall robustness, but connectivity is also a potential pathway for disaster spreading.

For example, in a recent paper entitled Systemic Risk and Stability in Financial Networks, one of their authors summarizes their findings as such: “We show that financial contagion exhibits a form of phase transition as interbank connections increase. As long as the magnitude and the number of negative shocks affecting financial institutions are sufficiently small, more ‘complete’ interbank claims enhance the stability of the system. However, beyond a certain point, such interconnections start to serve as a mechanism for propagation of shocks and lead to a more fragile financial system.”

### Failure Propagation

There are a few key parameters that will greatly affect this process of failure propagation within these complex networks. Firstly, how contagious is the phenomenon that is spreading? An important consideration here is whether this is being powered by some negative feedback loop. Secondly, how resistant are the nodes in the network to this phenomenon? Thirdly, we need to consider the topology to the network. Is it centralized or decentralized? Centralized networks are more susceptible to certain kinds of attack. Lastly, we need to also take into account whether this failure is being spread strategically or at random, as different network topologies exhibit different vulnerability characteristics depending on how random the failure is.

# Network Robustness

Robustness & resilience are often thought of in terms of a system's capacity to maintain functionality in the face of external perturbations. We see some extraordinary examples of this, ecological networks that persist despite extreme environmental changes. Communication networks like the internet can often deal with malfunction, errors, and attacks, without these local events leading to catastrophic global failures. But we also see the opposite where some small failure in say, a financial network can propagate to affect the whole system.

Trying to understand how and why this happens is the study of network robustness. Robustness can be correlated with connectivity in that connectivity enables system integration. Without connectivity parts to the system may become disconnected and disintegrated. If blood stops flowing to some part of the body, then it will become atrophied and waste away. Or if a child stops talking to their parents, then the family unit disintegrates through lack of communications. Thus, when we are talking about robustness & resilience, we are often asking what will happen to the network's overall connectivity and integration if we remove some components or connections, and equally, how will this failure then spread within the network system.

### Node Percolation

Failure can either be a cause of the network's nodes, where we are interested in what will happen if we remove a certain amount – this is called node percolation – but we can also talk about failure in terms of the removal of edges; edge percolation. And another key factor here is whether the attack is random or strategic. When we are talking about robustness with respect to the nodes in the network, then a key factor is the degree of distribution between the nodes.

The higher that degree distribution, meaning there will be more hubs, the more vulnerable it is to strategic attack. But if it is a random attack, then the degree distribution is not so important, as these hubs that are part of centralized networks will be particularly vulnerable to strategic attack. Thus, distributed networks will be robust to strategic attack but scale-free centralized networks are particularly susceptible. A strategic attack on large hubs will drastically reduce the number of connections within the system, increasing the average path length significantly, which is a key measure of the system's overall integration. This has been confirmed by empirical data from the Internet and World Wide Web which show robustness to random attack but are significantly affected by strategic attack due to the presence of major hubs in the network.

## Edge Percolation

Edge percolation is when connections fail or are removed. An important factor here is the degree of betweenness to the network. Betweenness measures the number of bridging edges that represent the critical, irreplaceable connections between one cluster and another. An example of this might be the Malacca Straits, a stretch of sea between the coast of Malaysia and Indonesia that connects maritime transport in Asia with the Middle East and Europe. Approximately 40 percent of the world's trade and 25 percent of all crude petroleum is thought to pass through this critical link in the global logistics network. This is an example of a bridging link that reduces the system's robustness and makes it much more susceptible to a strategic attack.

## Cascading Failure

Robustness does not just depend on if a node or edge will fail, but just as importantly what will happen when it fails. In other words, will the failure end there, or will it have a cascading effect as may often be the case? For example, failure in a power grid can result in other power stations becoming overloaded, allowing for the failure to propagate. When we are modeling robustness in terms of some external perturbation that propagates through the system, destroying links and nodes on its way, we want to ask how easily does it spread and what is the resistance to its spreading within the network.

One method for preventing failure propagation is through buffers and redundancy. When we engineer networks, these buffers are often artificially superimposed on the network. But when we look at the robustness in ecosystems, it is built into these networks in the form of diversity. Diversity is both a buffer in that the difference between nodes will present a barrier to complete contagion, and it is a form of redundancy as components are also similar; they can to a certain extent just replace other components.

## Diversity

Diversity in connections often comes in the form of weak ties, where the connections in a network can be divided into strong or weak, which is really a way of defining connections in terms of their frequency of interaction. In a social network, a strong tie is someone you interact with on a daily basis. Thus, strong ties are often between members of a cluster or clique, whereas a weak tie might be with someone you only talk to once every few months or once a year. Whereas strong ties may dominate a network in terms of quantity, weak ties are important in that they connect nodes from different clusters, making them bridges that add some diversity to the network which can be vital to its robustness if some subset of a network becomes infected. We could cite the phenomenon of 'groupthink' here, where a small social network or clique, often quickly converges to an agreed opinion on a subject without full consideration of different options. But by say, having some external consultant join the group, they would have a weak tie to add a diverse perspective and resist 'groupthink' from prevailing.

## Dynamics

Real-world networks are typically dynamic, where the network can adapt to some external perturbation. For example within the internet, routers have routing tables that keep track of how effective any path through the network is, and then update where it sends new packets based upon this. If we remove some set of edges, it will dynamically reroute packets to try and maintain network functionality, and this would be the same for a logistic network, a criminal network, or any other network with some kind of control system that allows it to adapt.

## Connectivity & Robustness

Connections enable integration and this is a key source of robustness as it bonds the system's components together. But connectivity can also be a pathway for disaster spreading. The most recent financial crisis might be a good example of this, where unknown linkages between complex financial instruments and institutions led to rapid contagion.

Thus, we need to be aware that every link in the network has a cost in terms of robustness. If it is not contributing to the system's integration and robustness, then it may be depleting from it as just another pathway for contagion spreading. We should be under no delusion that connectivity is always in some way a good thing. Research has shown that within certain settings connectivity can, up to a certain point, add to the system's robustness but beyond this hyper-connectivity can just be adding pathways for failures to propagate and overall fragility.

# Network Diffusion & Contagion

How something spreads across a network is a key question of interest when analyzing many different networks, the classical example of this being the diffusion of a disease through some population. But we might be talking about how the loss of one species in an ecosystem has an effect on others, the spread of financial contagion from one institution to another, or the spread of some information within a group of people. More formally, we call this spreading on a network, propagation or diffusion.

How diffusion happens and how long it takes is defined by a number of different parameters. Firstly, there is some infectiousness metric, where we are talking about how infectious the phenomenon that is spreading on the network is. A corollary to this is asking how resistant are the nodes to this contagion; which gives us a resistance parameter. Next, one needs to consider the topology to the network. Obviously, this diffusion is taking place along the connections within the network, meaning different structures to the connections and different degree distributions will be another defining factor when considering diffusion. Lastly, one needs to consider if this diffusion is taking place strategically or at random; this ties back to topology because some network topologies are more susceptible to strategic influence than others.

## Infectiousness

Infectiousness refers to something that is likely to spread or influence others in a rapid manner, irrespective of the type of network it is spreading on. If you hear about some important piece of news, you feel driven to tell others and that is infectiousness. It is like an outward force that is pushing the phenomena across local connections and out over the network. We may be able to quantify this in terms of money or how contagious a disease is or a number of other metrics, but we also need to ask how many nodes a given node can infect in any given time interval. A mosquito can only bite one other creature at a time but a person can broadcast a message to possibly millions of other people at any given instance, thus enabling a much more rapid contagion rate. Inversely, we need to consider how resistant the nodes in the network are to the spreading of this phenomenon.

Imagine trying to promote gay marriage in some conservative rural community. No matter how infectious your campaign is, it is unlikely to take off and this is due not to your failures but to the resistance of the other nodes in the network to this phenomenon. We could also add time to our model here, capturing how nodes may be affected for only a brief period of time before recovery; as would be the case with the spread of many diseases or some trend in fashion.

## Topology

The network's topology is a key consideration in understanding how something is likely to spread across it, the primary factor here being simply the overall degree of connectivity to the network. Obviously the more connected it is the faster something should spread across it, but also we would need to look at the average shortest path to get an idea of how many edges any phenomenon would have to traverse in order to affect the whole system.

For example, modern broadcast media has arisen hand-in-hand with the modern nation state, as it is only through these centralized hubs that uniform information can be rapidly disseminated to a large population, and thus a key component in creating a sense of nation culture and cohesion. Without these centralized hubs, diffusion can be a lot slower and become heterogeneous.

## Strategic Attack

We also need to ask whether this dissemination is random or strategic. That is, whether there is some logic behind the promotion and dissemination aimed at strategically affecting nodes that have a high degree of connectivity, and thus enabling a more rapid diffusion. Many forms of diffusion can be modeled as random. A virus has no logic telling it to attack creatures that have lots of physical contact with others.

We might say the same of financial contagion. Toxic assets do not themselves choose where to end up in the network and which nodes to affect most. These are factors that are defined by other dynamics. But some diffusion processes are strategic. For example, military strategy is often specifically designed to attack a critical node in an opposition's military or infrastructure network in the hope that this shock to a critical node will then propagate to its many dependent nodes, and thus have a greater effect than simply choosing to attack any node at random.

## Contagion

Complex contagion is the process in which multiple sources of exposure to a phenomenon are required before an individual adopts the change in its state. An example of this might be the adoption of some new technology or innovation which is costly, especially for early adopters, but less so for those who wait. We can then model this as a form of complex contagion, asking how many other nodes need to adopt the innovation before a given node will do likewise.

There might also be two competing events propagating across the network. An example of this could be trying to model how an individual will vote for two different candidates in an election based upon the social network they are a part of. We would then be defining some variable as to how many of the node's neighbors need to vote for a particular party before they would cast their vote for the same party. These complex models have many interacting parts. Thus, there will be tipping points, as a node will not do anything until a threshold value is met. There is feedback, as when the node changes its state it will affect the choices of others around it also. All of this means that this more complex form of contagion is nonlinear, with the possibility of exponential cascades forming.

Real-world diffusion across something like a social network is a complex process that may require multiple network models, that is, allowing the network model to have multiple different connections between nodes; in order to capture how different types of connections and networks interact to enable or resist the diffusion of some phenomena. Within the voting example, we might have to take into account economic factors and other social relations in order to capture the true dynamics at play.

# Complex Contagion

Complex contagion is the phenomenon in social networks in which multiple sources of exposure to some alteration are required before an individual adopts the change of behavior. This differs from simple contagion in that, it may not be possible for the innovation to spread after only one incident of contact with an infected neighbor.

The spread of complex contagion across a network of people may depend on many social and economic factors; for instance, how many of one's friends adopt the new idea as well as how strongly they actually influence the individual. In complex contagion, the probability of adopting a behavior, or an idea, varies with the extent of exposure. As an example a person might not respond when they see a piece of information on one social media site, but when they see it on another or a third this may trigger them to have greater belief in that piece of information and start to share it.

When we allow for this more complex form of contagion we now have to start to take into account different sources of contagion that may be conflicting. The spreading of propaganda may be an example of this, within a very simple homogeneous scenario where we have just one national broadcaster we will have a relatively simple contagion process, with just one single message being propagated. But in a more complex setting with multiple channels, there may be conflicting messages and we have to understand the network of interacting messages that people are receiving and also the significance that they ascribe to those different channels.

# Network Effects

The network effect arises when users gain value from others using the same network. The more people that join the more value for everyone else, this dynamic is driven by positive externalities and feedback loops. For example, when a person chooses to learn a particular language they are not just generating value for themselves but also some of the value is being externalized to everyone else who is using that network, as they now have more communications options available to them due to this positive externality.

The network effect gives us what is called Metcalfe's law which suggests that the value of a network is proportional to the square of the number of users of that network. because of all of these positive externalities the system as a whole now has a value greater than it individuals. With the network effect, people will not only adopt a phenomenon based upon its value in isolation but also on an assessment of how many others will also adopt the phenomenon; we choose to go to a party or some gathering only if we think others will also go.

Thus, expectation becomes very important, people not only have to value something but they have to expect that others will also adopt it. In this way, expectation can be a very high leverage point with respect to diffusion on social networks. The network effect is also notorious for creating lock-in because there is so much value created by everyone simply using the same network, this creates a strong force towards convergence – everyone using the one network at the expense of all others. We can see this with the dominance of English as a global language and the decline of many other smaller languages.

## Examples

A good example of the network effect would be a language, the value of some language is relative to the number of other users of that language, the more people that adopt that language the more valuable it will be. People learn English, Spanish and Chinese as a second language not because those languages are in anyway better than others, but simply because billions of people speak these languages giving them a powerful network effect and lots of value.

The network effect may be seen in the formation and spreading of many phenomena within social networks, such as the spreading of some fashion. As is invariably the case with positive feedback, the network effect results in exponential growth, tipping points, and cascades. Network effects may also be seen in stock markets and derivatives exchanges. Market liquidity is a major determinant of transaction cost in the sale or purchase of a security. As the number of buyers and sellers on an exchange increases, liquidity increases, and transaction costs decrease. This then attracts a larger number of buyers and sellers to the exchange. Thus we can see the positive feedback loop that is behind the network effect.

## Tipping Points

This network effect may give the diffusion process a strong tipping point, because below a certain level of people adopting that phenomenon the value is very low, we might say sublinear. Adopting some radical new fashion when no one else has will come at a great social cost, but doing it when everyone else has will come at a much greater value.

In the initial phase of a network's formation, due to the limited number of nodes and connections in the network, the value of joining that network may, in fact, be negative because of the opportunity cost. Joining this network may well exclude you from joining another more mature network that already has a lot of network value. For example, if you choose to adopt a Linux operating system, you will be limiting your capacity to interoperate with over one billion users of Windows. Thus, in terms of opportunity cost, you are actually having to pay to be part of this burgeoning Linux network, and the same would be true for a social network, digital currencies and many other types of networks that have not reached a critical mass.

These early adopters are typically special interest users that particularly care about this service and are prepared to pay the opportunity cost. Thus the pioneers of some new phenomena – whether we are talking about a new political opinion, a new social movement or a new style – these first adopters will have to be very committed putting in a lot of resources and getting little out, but if the phenomenon does spread then the network effect will take hold, there will be a snowball effect due to the positive externalities, there will be some tipping point or phase transition where it rapidly goes from a fringe activity to a mainstream phenomenon and the course of least resistance.

## Multiplex Networks

A multiplex network is a network model composed of a multiplicity of overlapping networks that capture the different types of connections between nodes within the network. When analyzing a large system like a metropolitan economy, a corporation or the global commodities market, what we are dealing with is a network that is embedded within many other networks. A metropolitan area will be a complex system where economic interactions are embedded within socio-political networks, transportation and geospatial networks and financial networks, in order to capture all of these overlapping networks it is optimal to use a multiplex network model. In a multiplex network, each type of interaction between the nodes is described by a single layer network and the different layers of networks describe the different modes of interaction.

With a multiplex network we can try and capture how these different connections interact and affect each other. This is a much more realistic picture that lies behind many phenomena and a lot more faithful to one of the basic premises of complexity theory, that is that many phenomena are in fact the product of a multiplicity of nonlinear interacting forces.

As an example we might think about the recent uprisings in Egypt. When we first look at this phenomenon we would consider it political in nature and start analysing the political network, but research has shown a robust correlation between spikes in the price of basic foods and the occurrence of these riots, thus these events are an emergent phenomenon of different interacting networks, social, political and economic all putting stress on the social system. In this situation it would be of use to use a multiplex network to try and model the overall dynamic.

Phenomena like this are very complex they are embedded within many different overlapping networks, simply modeling one of these networks can only ever give a partial insight, thus a full representation necessitates multiplex network modeling.